week5: The Euler-Lagrange Equations of Classical Mechanics: Derivation for N Particles

We now want to transfer the considerations from `week3.pdf` to N-particle systems. We first describe the general setting and then give two examples, N charged balls in a bowl and the double pendulum, which this setting is intended to cover, so to speak. At the end we summarize the results in Theorem 2.3.2.

We thus consider N particles with masses $m_1, m_2, \cdots, m_N$ and position vectors $\vec{x}_1, \vec{x}_2, \cdots$ $\cdots, \vec{x}_N \in \mathbb{R}^3$, which move under the influence of a potential

$$V \;=\; V(\vec{x}_1, \cdots, \vec{x}_N) \tag{1}$$

Through the potential V a force

$$\vec{F}_i \;=\; -\nabla_{\vec{x}_i} V \tag{2}$$

is exerted on the i-th particle.

Let the motion of the N particles not be completely free, but subject to mechanical restrictions. Let these constraints be given by the n (non-redundant, linearly independent) equations

$$\begin{aligned} g_1(\vec{x}_1, \cdots, \vec{x}_N) &= 0 \\ &\vdots \\ g_n(\vec{x}_1, \cdots, \vec{x}_N) &= 0 \end{aligned} \tag{3}$$

If we collect the position vectors of the N particles into the vector

$$\vec{x} \;:=\; (\vec{x}_1, \cdots, \vec{x}_N) \;\in\; \mathbb{R}^{3N}$$

then the motion takes place on an

$$f \;:=\; 3N \,-\, n \tag{4}$$

dimensional surface $\mathcal{F}$, or submanifold, of $\mathbb{R}^{3N}$, with

$$\mathcal{F} \;:=\; \left\{ \vec{x} \in \mathbb{R}^{3N} \mid g_j(\vec{x}_1, \cdots, \vec{x}_N) = 0 \;\forall\, j = 1, ..., n \right\} \;\subset\; \mathbb{R}^{3N} \tag{5}$$

Every parametrization of $\mathcal{F}$ by f independent coordinates

$$q \;=\; (q_1, \cdots, q_f) \tag{6}$$

defines the generalized coordinates for the given mechanical problem. That is, the Cartesian coordinates $\vec{x}_1, ..., \vec{x}_N$ are to be expressed in terms of $q_1, ..., q_f$,

$$\begin{aligned} \vec{x}_1 &= \vec{x}_1(q_1, q_2, \cdots, q_f) = \vec{x}_1(q) \\ &\vdots \\ \vec{x}_N &= \vec{x}_N(q_1, q_2, \cdots, q_f) = \vec{x}_N(q) \end{aligned} \tag{7}$$

such that the constraints (3) are automatically satisfied:

$$\begin{aligned} g_1\big(\vec{x}_1(q), \cdots, \vec{x}_N(q)\big) &= 0 \\ &\vdots \\ g_n\big(\vec{x}_1(q), \cdots, \vec{x}_N(q)\big) &= 0 \end{aligned} \tag{8}$$

Thus $\mathcal{F}$ is parametrized by the generalized coordinates $q = (q_1, \cdots, q_f)$, and the vectors

$$\tfrac{\partial \vec{x}}{\partial q_1}\,, \quad \cdots \quad, \tfrac{\partial \vec{x}}{\partial q_f} \tag{9}$$

are linearly independent tangent vectors to $\mathcal{F}$. We can extend these vectors to a basis of $\mathbb{R}^{3N}$ by also including the $3N - f = 3N - (3N - n) = n$ gradients

$$\nabla_{\vec{x}}\, g_1\,, \ \cdots\ , \nabla_{\vec{x}}\, g_n \tag{10}$$

That is,

$$\mathcal{B} = \big\{ \tfrac{\partial \vec{x}}{\partial q_1}\,, \cdots, \tfrac{\partial \vec{x}}{\partial q_f}\,, \nabla_{\vec{x}} g_1\,, \cdots, \nabla_{\vec{x}} g_n \big\} \tag{11}$$

is a basis of $\mathbb{R}^{3N}$, and the gradients are perpendicular to the tangent vectors, since: differentiating the k-th equation of (8) with respect to q_j gives

$$\begin{aligned} 0 &= \tfrac{\partial}{\partial q_j}\, g_k\big(\vec{x}_1(q), \cdots, \vec{x}_N(q)\big) \\ &= \sum_{i=1}^{N} \nabla_{\vec{x}_i}\, g_k\big(\vec{x}_1(q), \cdots, \vec{x}_N(q)\big)\ \tfrac{\partial \vec{x}_i}{\partial q_j} \\ &= \nabla_{\vec{x}}\, g_k\ \tfrac{\partial \vec{x}}{\partial q_j} \end{aligned} \tag{12}$$

In the second line of (12) we have a sum of scalar products in $\mathbb{R}^3$; the last line of (12) is given by a scalar product in $\mathbb{R}^{3N}$.

Before we write down the equations of motion, we want to give two examples for which this setting is, roughly speaking, intended.

Example 1: N electrostatically charged balls in a spherical bowl under the influence of gravity: In this case the potential would be given by

$$V(\vec{x}_1, \cdots, \vec{x}_N) = \sum_{i=1}^{N} mg\, z_i \ + \ \tfrac{1}{4\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ i<j}}^{N} \tfrac{Q_i Q_j}{\|\vec{x}_i - \vec{x}_j\|}$$

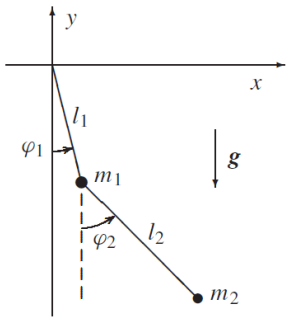
where $Q_1, ..., Q_N$ are the charges of the balls and, say, $\vec{x}_i = (x_i, y_i, z_i)$. We have $n = N$ constraints, given by the equations

$$x_i^2 + y_i^2 + z_i^2 - R^2 = 0 \qquad \forall\, i = 1, \cdots, N$$

where R is the radius of the bowl. The surface $\mathcal{F}$ would then be $f = 3N - N = 2N$ dimensional, and as a parametrization we could simply take the angles of the spherical coordinates, respectively,

$$(q_1, q_2, \cdots, q_{2N-1}, q_{2N}) = (\varphi_1, \theta_1, \cdots, \varphi_N, \theta_N)\,.$$

Example 2: The planar double pendulum under the influence of gravity: it looks as follows,



Here we thus have a planar pendulum, not a spherical one in $\mathbb{R}^3$ as last week, with

$$\begin{aligned} \vec{x}_1 &= (x_1, y_1) \in \mathbb{R}^2 \\ \vec{x}_2 &= (x_2, y_2) \in \mathbb{R}^2 \end{aligned}$$

with the $n = 2$ constraints

$$\begin{aligned} x_1^2 + y_1^2 - \ell_1^2 &= 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - \ell_2^2 &= 0 \end{aligned}$$

where ℓ_1 and ℓ_2 are, respectively, the pendulum lengths. The potential is

$$V(\vec{x}_1, \vec{x}_2) = m_1 g\, y_1 + m_2 g\, y_2$$

The surface $\mathcal{F}$ is $f = 2N - n = 4 - 2 = 2$ dimensional, and as generalized coordinates one could take the angles φ_1 and φ_2 shown in the figure.

Back to the general setting, equations of motion:

What do the equations of motion look like? Forces generated by the potential V act on the N-particle system, as well as constraint forces:

$$m_i\, \ddot{\vec{x}}_i = \vec{F}_i + \vec{Z}_i$$

or

$$m_i\, \ddot{\vec{x}}_i + \nabla_{\vec{x}_i} V = \vec{Z}_i \tag{13}$$

If we collect the constraint forces $\vec{Z}_i$ into a vector in $\mathbb{R}^{3N}$,

$$\vec{Z} \;:=\; (\vec{Z}_1, \cdots, \vec{Z}_N) \tag{14}$$

then $\vec{Z}$ must have no tangential component with respect to $\mathcal{F}$. That is, using the basis $\mathcal{B}$ from (11),

$$\vec{Z} \;=\; \sum_{k=1}^{n} \lambda_k \, \nabla_{\vec{x}} \, g_k \tag{15}$$

for certain functions $\lambda_k = \lambda_k(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})$. The constraint force for the i-th particle is then given by

$$\vec{Z}_i \;=\; \sum_{k=1}^{n} \lambda_k \, \nabla_{\vec{x}_i} \, g_k \tag{16}$$

and the equations of motion read

$$m_i \, \ddot{\vec{x}}_i \;+\; \nabla_{\vec{x}_i} V \;=\; \sum_{k=1}^{n} \lambda_k \, \nabla_{\vec{x}_i} g_k \tag{17}$$

We multiply (17) by $\frac{\partial \vec{x}_i}{\partial q_j}$ and sum over i

$$\begin{aligned}
\sum_{i=1}^{N} m_i \, \ddot{\vec{x}}_i \, \tfrac{\partial \vec{x}_i}{\partial q_j} \;+\; \sum_{i=1}^{N} \nabla_{\vec{x}_i} V \, \tfrac{\partial \vec{x}_i}{\partial q_j} \;&=\; \sum_{k=1}^{n} \lambda_k \, \sum_{i=1}^{N} \nabla_{\vec{x}_i} \, g_k \, \tfrac{\partial \vec{x}_i}{\partial q_j} \\
\Leftrightarrow \qquad \sum_{i=1}^{N} m_i \, \ddot{\vec{x}}_i \, \tfrac{\partial \vec{x}_i}{\partial q_j} \;+\; \tfrac{\partial V}{\partial q_j} \;&=\; \sum_{k=1}^{n} \lambda_k \, \underbrace{\nabla_{\vec{x}} \, g_k \, \tfrac{\partial \vec{x}}{\partial q_j}}_{=\,0 \text{ by (12)}} \\
\Leftrightarrow \qquad \sum_{i=1}^{N} m_i \, \ddot{\vec{x}}_i \, \tfrac{\partial \vec{x}_i}{\partial q_j} \;+\; \tfrac{\partial V}{\partial q_j} \;&=\; 0
\end{aligned} \tag{18}$$

Because of

$$\dot{\vec{x}}_i \;=\; \sum_{j=1}^{f} \tfrac{\partial \vec{x}_i}{\partial q_j} \, \dot{q}_j \tag{19}$$

we can again regard the kinetic energy T (as in `week3.pdf` with $u, v, \dot{u}, \dot{v}$) as a function of the q_j and the $\dot{q}_j$,

$$\begin{aligned}
T \;=\; T(q, \dot{q}) \;&=\; \sum_{i=1}^{N} \tfrac{m_i}{2} \, \dot{\vec{x}}_i^{\,2} \\
&\overset{(19)}{=} \; \sum_{j,k=1}^{f} \dot{q}_j \dot{q}_k \, \sum_{i=1}^{N} \tfrac{m_i}{2} \, \tfrac{\partial \vec{x}_i}{\partial q_j} \, \tfrac{\partial \vec{x}_i}{\partial q_k} \\
&=\; \sum_{j,k=1}^{f} \dot{q}_j \dot{q}_k \, t_{jk}(q)
\end{aligned} \tag{20}$$

with

$$t_{jk}(q) \;:=\; \sum_{i=1}^{N} \tfrac{m_i}{2}\, \tfrac{\partial \vec{x}_i}{\partial q_j} \tfrac{\partial \vec{x}_i}{\partial q_k} \tag{21}$$

The following statement then holds, which we can copy almost verbatim from Theorem 2.1.1 in `week3.pdf`:

Theorem 2.3.1: Let $\vec{x} = \vec{x}(q)$ be a parametrization of $\mathcal{F}$, $\vec{x}_t = \vec{x}(q_t)$, and let the function $T = T(q, \dot{q})$ be given by (20). Then

$$\big(\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_j} \,-\, \tfrac{\partial}{\partial q_j}\big)\big\{\tfrac{m_i}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \;=\; m_i \ddot{\vec{x}}_i\, \tfrac{\partial \vec{x}_i}{\partial q_j} \tag{22}$$

for all $j = 1, 2, ..., f$. In particular:

$$\sum_{i=1}^{N} m_i\, \ddot{\vec{x}}_i\, \tfrac{\partial \vec{x}_i}{\partial q_j} \;=\; \tfrac{d}{dt}\tfrac{\partial T}{\partial \dot{q}_j} \,-\, \tfrac{\partial T}{\partial q_j} \tag{23}$$

with the kinetic energy

$$T \;=\; \sum_{i=1}^{N} \tfrac{m_i}{2}\, \dot{\vec{x}}_i^{\,2} \;=\; \sum_{j,k=1}^{f} \dot{q}_j \dot{q}_k\, t_{jk}(q) \tag{24}$$

and $t_{jk}(q)$ as given in (21).

Proof: With

$$\dot{\vec{x}}_i \;=\; \sum_{k=1}^{f} \tfrac{\partial \vec{x}_i}{\partial q_k}\, \dot{q}_k \tag{25}$$

we obtain

$$\tfrac{\partial}{\partial \dot{q}_j}\big\{\tfrac{1}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \;=\; \dot{\vec{x}}_i\, \tfrac{\partial \dot{\vec{x}}_i}{\partial \dot{q}_j} \;\overset{(25)}{=}\; \dot{\vec{x}}_i\, \tfrac{\partial \vec{x}_i}{\partial q_j}$$

and hence

$$\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_j}\big\{\tfrac{1}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \;=\; \ddot{\vec{x}}_i\, \tfrac{\partial \vec{x}_i}{\partial q_j} \;+\; \dot{\vec{x}}_i\, \tfrac{d}{dt}\tfrac{\partial \vec{x}_i}{\partial q_j}$$

On the other hand,

$$\tfrac{\partial}{\partial q_j}\big\{\tfrac{1}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \;=\; \dot{\vec{x}}_i\, \tfrac{\partial \dot{\vec{x}}_i}{\partial q_j} \;\overset{(25)}{=}\; \dot{\vec{x}}_i \sum_{k=1}^{f} \tfrac{\partial^2 \vec{x}_i}{\partial q_j \partial q_k}\, \dot{q}_k \;=\; \dot{\vec{x}}_i \sum_{k=1}^{f} \tfrac{\partial}{\partial q_k}\big(\tfrac{\partial \vec{x}_i}{\partial q_j}\big)\, \dot{q}_k \;=\; \dot{\vec{x}}_i\, \tfrac{d}{dt}\tfrac{\partial \vec{x}_i}{\partial q_j}$$

Hence,

$$\tfrac{d}{dt}\tfrac{\partial}{\partial \dot{q}_j}\big\{\tfrac{1}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \,-\, \tfrac{\partial}{\partial q_j}\big\{\tfrac{1}{2}\,\dot{\vec{x}}_i^{\,2}\big\} \;=\; \ddot{\vec{x}}_i\, \tfrac{\partial \vec{x}_i}{\partial q_j}$$

and the theorem is proven. ■

Using Theorem 2.3.1 we can thus write the system (18) as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \tag{26}$$

or

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{27}$$

Since, by assumption, V depends only on the $\vec{x}_i$ but not on the $\dot{\vec{x}}_i$, we have $\frac{\partial V}{\partial \dot{q}_j} = 0$, and we can finally write

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0 \tag{28}$$

We again summarize our results (analogously to Theorem 2.1.2 in `week3.pdf`) in a theorem.

Theorem 2.3.2: Let an N-particle system move under the influence of a potential $V = V(\vec{x}_1, \cdots, \vec{x}_N)$. Through this potential the force

$$\vec{F}_i = -\nabla_{\vec{x}_i} V(\vec{x})$$

is exerted on the i-th particle. Furthermore, suppose the particle system is subject to mechanical constraints such that its motion can take place only within an f-dimensional surface

$$\mathcal{F} = \left\{ \vec{x} = (\vec{x}_1, \cdots, \vec{x}_N) \in \mathbb{R}^{3N} \;\middle|\; g_j(\vec{x}_1, \cdots, \vec{x}_N) = 0 \quad \forall\, j = 1, ..., n \right\} \subset \mathbb{R}^{3N}$$

Then:

a) The trajectories $\vec{x}_i(t) = \vec{x}_i(q_t)$ satisfy the equations of motion

$$m_i \ddot{\vec{x}}_i + \nabla_{\vec{x}_i} V(\vec{x}) = \vec{Z}_i$$

with the constraint forces

$$\vec{Z}_i = \sum_{k=1}^{n} \lambda_k(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})\, \nabla_{\vec{x}_i}\, g_k(\vec{x})$$

The constraint forces $\vec{Z}_i$ are real forces, not merely a mathematical device.

b) If we define the Lagrangian L by

$$L := T - V = \sum_{i=1}^{N} \frac{m_i}{2} \dot{\vec{x}}_i^2 - V(\vec{x}_1, \cdots, \vec{x}_N) \tag{29}$$

then the trajectories of the generalized coordinates $q_1(t), ..., q_f(t)$, and hence also the trajectories in Cartesian coordinates $\vec{x}_i(t) = \vec{x}_i\big(q_1(t), \cdots, q_f(t)\big)$, can be obtained from the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \tag{30}$$

Remarks:

1) To set up and solve the Euler-Lagrange equations (30), i.e. to compute the trajectories, knowledge of the constraint forces $\vec{Z}_i$ is not required. The relevant part of the theorem is thus typically part (b); part (a) is 'just for information', so to speak, just for a better understanding.

2) The parameters $q_t = \big(q_1(t), \cdots, q_f(t)\big)$ (or any choice of such parameters) that ensure the constraints

$$\vec{x} \;=\; \vec{x}(q_t) \quad \in \quad \mathcal{F}$$

are called **generalized coordinates** for the given mechanical problem.

3) The theorem also holds for the case $\mathcal{F} = \mathbb{R}^{3N}$, i.e. for the case that there are no constraints at all. It then states that the equations of motion, in arbitrary coordinates, are always of the form (30). Whereas, for instance, $m\,\ddot{\vec{x}} = 0$ or $m\ddot{x} = 0$ and $m\ddot{y} = 0$ only look like this in Cartesian coordinates – in polar coordinates $m\ddot{r} = 0$ and $m\ddot{\varphi} = 0$ no longer hold, but the equations are somewhat different – the form of (30) is always the same, independent of which particular parametrization was chosen.