week4: Examples for Chapter 2.1: The Euler-Lagrange equations of classical mechanics, 1 particle

We want to illustrate the results of last week on two examples. In the first example, we consider the force-free motion in $\mathbb{R}^3$ in spherical coordinates. We thus do exactly the same as we did already on exercise sheet 3, but now with spherical coordinates in $\mathbb{R}^3$ instead of polar coordinates in $\mathbb{R}^2$. The formulae we derive there then will be used in the second example where we discuss the motion of a point mass or of a ball in a spherical bowl. In the textbooks or on the internet this typically can be found under the term 'spherical pendulum'.

Example 1: Force-free motion of a particle in $\mathbb{R}^3$ in spherical coordinates

Spherical coordinates in $\mathbb{R}^3$ are given by the parametrization

$$\vec{x} \;=\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; r \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix} \tag{1}$$

with

$$\begin{aligned} r &\in [0, +\infty) \\ \varphi &\in [0, 2\pi) \\ \theta &\in [0, \pi] \end{aligned} \tag{2}$$

We introduce the following basis vectors:

$$\vec{e}_r \;:=\; \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix}, \quad \vec{e}_\varphi \;:=\; \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \vec{e}_\theta \;:=\; \begin{pmatrix} \cos\varphi \, \cos\theta \\ \sin\varphi \, \cos\theta \\ -\sin\theta \end{pmatrix} \tag{3}$$

Then

$$\|\vec{e}_r\|^2 \;=\; \|\vec{e}_\varphi\|^2 \;=\; \|\vec{e}_\theta\|^2 \;=\; 1$$

and

$$\vec{e}_r \, \vec{e}_\varphi \;=\; \vec{e}_r \, \vec{e}_\theta \;=\; \vec{e}_\varphi \, \vec{e}_\theta \;=\; 0$$

that is, the vectors $\{\vec{e}_r, \vec{e}_\varphi, \vec{e}_\theta\}$ form an orthonormal basis. We now consider a trajectory $\vec{x}_t \in \mathbb{R}^3$ and parametrize $\vec{x}_t$ using spherical coordinates $(r_t, \varphi_t, \theta_t)$. We then have

$$\vec{x}_t \;=\; \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \;=\; r_t \begin{pmatrix} \cos\varphi_t \, \sin\theta_t \\ \sin\varphi_t \, \sin\theta_t \\ \cos\theta_t \end{pmatrix} \;=\; r_t \, \vec{e}_r \tag{4}$$

Instead of $\vec{e}_{r_t}$ we simply write $\vec{e}_r$ in all what follows, the time-dependence of the parameters is thus implicitely understood. Let's start by computing the time derivatives of the basis vectors. We have

$$\dot{\vec{e}}_r \;=\; \dot\varphi \begin{pmatrix} -\sin\varphi\,\sin\theta \\ \cos\varphi\,\sin\theta \\ 0 \end{pmatrix} \;+\; \dot\theta \begin{pmatrix} \cos\varphi\,\cos\theta \\ \sin\varphi\,\cos\theta \\ -\sin\theta \end{pmatrix} \;=\; \dot\varphi\,\sin\theta\,\vec{e}_\varphi \;+\; \dot\theta\,\vec{e}_\theta$$

$$\dot{\vec{e}}_\varphi \;=\; \dot\varphi \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} \;=\; \dot\varphi\,\big\{-\sin\theta\,\vec{e}_r \;-\; \cos\theta\,\vec{e}_\theta\big\}$$

$$\dot{\vec{e}}_\theta \;=\; \dot\varphi\,\cos\theta \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \;-\; \dot\theta \begin{pmatrix} \cos\varphi\,\sin\theta \\ \sin\varphi\,\sin\theta \\ \cos\theta \end{pmatrix} \;=\; \dot\varphi\,\cos\theta\,\vec{e}_\varphi \;-\; \dot\theta\,\vec{e}_r$$

Summarized,

$$\begin{aligned} \dot{\vec{e}}_r &= \dot\varphi\,\sin\theta\,\vec{e}_\varphi \;+\; \dot\theta\,\vec{e}_\theta \\ \dot{\vec{e}}_\varphi &= -\dot\varphi\,\sin\theta\,\vec{e}_r \;-\; \dot\varphi\,\cos\theta\,\vec{e}_\theta \\ \dot{\vec{e}}_\theta &= \dot\varphi\,\cos\theta\,\vec{e}_\varphi \;-\; \dot\theta\,\vec{e}_r \end{aligned} \tag{5}$$

We obtain

$$\begin{aligned} \vec{x} &= r\,\vec{e}_r \\ \dot{\vec{x}} &= \dot r\,\vec{e}_r \;+\; r\,\dot{\vec{e}}_r \\ \ddot{\vec{x}} &= \ddot r\,\vec{e}_r \;+\; 2\,\dot r\,\dot{\vec{e}}_r \;+\; r\,\ddot{\vec{e}}_r \end{aligned}$$

and further

$$\begin{aligned} \ddot{\vec{x}} &= \ddot r\,\vec{e}_r \;+\; 2\,\dot r\,\dot{\vec{e}}_r \;+\; r\,\ddot{\vec{e}}_r \\ &= \ddot r\,\vec{e}_r \;+\; 2\,\dot r\,\big\{\dot\varphi\,\sin\theta\,\vec{e}_\varphi \;+\; \dot\theta\,\vec{e}_\theta\big\} \;+\; r\,\tfrac{d}{dt}\big\{\dot\varphi\,\sin\theta\,\vec{e}_\varphi \;+\; \dot\theta\,\vec{e}_\theta\big\} \\ &= \ddot r\,\vec{e}_r \;+\; 2\,\dot r\,\dot\varphi\,\sin\theta\,\vec{e}_\varphi \;+ 2\,\dot r\,\dot\theta\,\vec{e}_\theta \\ &\quad + r\Big\{\,\ddot\varphi\,\sin\theta\,\vec{e}_\varphi \;+\; \dot\varphi\,\dot\theta\,\cos\theta\,\vec{e}_\varphi \;+\; \dot\varphi\,\sin\theta\,\dot{\vec{e}}_\varphi \;+\; \ddot\theta\,\vec{e}_\theta \;+\; \dot\theta\,\dot{\vec{e}}_\theta\,\Big\} \\ &= \ddot r\,\vec{e}_r \;+\; \big\{2\,\dot r\,\dot\varphi\,\sin\theta \;+\; r\,\ddot\varphi\,\sin\theta \;+\; r\,\dot\varphi\,\dot\theta\,\cos\theta\big\}\,\vec{e}_\varphi \;+\; \big\{2\,\dot r\,\dot\theta \;+\; r\,\ddot\theta\,\big\}\vec{e}_\theta \\ &\quad + r\,\dot\varphi\,\sin\theta\,\dot{\vec{e}}_\varphi \;+\; r\,\dot\theta\,\dot{\vec{e}}_\theta \\ &= \ddot r\,\vec{e}_r \;+\; \big\{2\,\dot r\,\dot\varphi\,\sin\theta \;+\; r\,\ddot\varphi\,\sin\theta \;+\; r\,\dot\varphi\,\dot\theta\,\cos\theta\big\}\,\vec{e}_\varphi \;+\; \big\{2\,\dot r\,\dot\theta \;+\; r\,\ddot\theta\,\big\}\vec{e}_\theta \\ &\quad + r\,\dot\varphi\,\sin\theta\,\big[-\dot\varphi\,\sin\theta\,\vec{e}_r \;-\; \dot\varphi\,\cos\theta\,\vec{e}_\theta\big] \;+\; r\,\dot\theta\,\big[\dot\varphi\,\cos\theta\,\vec{e}_\varphi \;-\; \dot\theta\,\vec{e}_r\big] \end{aligned}$$

We collect the contributions for each basis vector $\{\vec{e}_r, \vec{e}_\varphi, \vec{e}_\theta\}$ and thus obtain the following

Lemma 2.1.3: In spherical coordinates we have:

$$\begin{aligned} \ddot{\vec{x}} \;=\; & \big\{\ddot r \;-\; r\,\dot\varphi^2\sin^2\theta \;-\; r\,\dot\theta^2\big\}\,\vec{e}_r \\ & + \big\{2\,\dot r\,\dot\varphi\,\sin\theta \;+\; r\,\ddot\varphi\,\sin\theta \;+\; 2\,r\,\dot\varphi\,\dot\theta\,\cos\theta\big\}\,\vec{e}_\varphi \\ & + \big\{2\,\dot r\,\dot\theta \;+\; r\,\ddot\theta \;-\; r\,\dot\varphi^2\sin\theta\cos\theta\big\}\,\vec{e}_\theta \end{aligned} \tag{6}$$

So: if we have force-free motion, the equation $m\,\ddot{\vec{x}} = 0$ holds, and by Lemma 2.1.3 this is equivalent to

$$\begin{aligned} \ddot{r} \;-\; r\,\dot{\varphi}^2 \sin^2\theta \;-\; r\,\dot{\theta}^2 &= 0 \\ 2\,\dot{r}\,\dot{\varphi}\,\sin\theta \;+\; r\,\ddot{\varphi}\,\sin\theta \;+\; 2\,r\,\dot{\varphi}\,\dot{\theta}\,\cos\theta &= 0 \\ 2\,\dot{r}\,\dot{\theta} \;+\; r\,\ddot{\theta} \;-\; r\,\dot{\varphi}^2 \sin\theta\cos\theta &= 0 \end{aligned} \tag{7}$$

Using the Lagrangian formalism, this result can be derived considerably faster: the Lagrangian L is given by

$$\begin{aligned} L &= \text{kinetic energy} \;-\; \text{potential energy} \\ &= T \;-\; V \\ &\overset{V=0}{=} T \\ &= \tfrac{m}{2}\,\dot{\vec{x}}^2 \end{aligned} \tag{8}$$

We have

$$\begin{aligned} \dot{\vec{x}} &= \dot{r}\,\vec{e}_r \;+\; r\,\dot{\vec{e}}_r \\ &\overset{(5)}{=} \dot{r}\,\vec{e}_r \;+\; r\,\big\{\,\dot{\varphi}\,\sin\theta\,\vec{e}_\varphi \;+\; \dot{\theta}\,\vec{e}_\theta\,\big\} \end{aligned}$$

and since the vectors $\{\vec{e}_r, \vec{e}_\varphi, \vec{e}_\theta\}$ form an orthonormal basis, it immediately follows that

$$L \;=\; \tfrac{m}{2}\,\dot{\vec{x}}^2 \;=\; \tfrac{m}{2}\Big\{\,\dot{r}^2 \;+\; r^2\dot{\varphi}^2\sin^2\theta \;+\; r^2\dot{\theta}^2\,\Big\} \tag{9}$$

The Euler-Lagrange equations read

$$\begin{aligned} \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{r}} \;-\; \tfrac{\partial L}{\partial r} &= 0 \\ \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\varphi}} \;-\; \tfrac{\partial L}{\partial \varphi} &= 0 \\ \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\theta}} \;-\; \tfrac{\partial L}{\partial \theta} &= 0 \end{aligned} \tag{10}$$

We can divide out the $m/2$, and the system (10) is then equivalent to

$$\begin{aligned} \tfrac{d}{dt}\big\{\,2\,\dot{r}\,\big\} \;-\; 2\,r\dot{\varphi}^2\sin^2\theta \;-\; 2\,r\dot{\theta}^2 &= 0 \\ \tfrac{d}{dt}\big\{\,2\,r^2\dot{\varphi}\sin^2\theta\,\big\} \;-\; 0 &= 0 \\ \tfrac{d}{dt}\big\{\,2\,r^2\dot{\theta}\,\big\} \;-\; 2\,r^2\dot{\varphi}^2\sin\theta\cos\theta &= 0 \end{aligned}$$

or

$$\begin{aligned} \ddot{r} \;-\; r\dot{\varphi}^2\sin^2\theta \;-\; r\dot{\theta}^2 &= 0 \\ \tfrac{d}{dt}\big\{\,r^2\dot{\varphi}\sin^2\theta\,\big\} &= 0 \\ \tfrac{d}{dt}\big\{\,r^2\dot{\theta}\,\big\} \;-\; r^2\dot{\varphi}^2\sin\theta\cos\theta &= 0 \end{aligned} \tag{11}$$

The first equation of (11) is exactly identical to the first equation of (7). If we carry out the time derivative in the second equation of (11), we obtain

$$
\begin{aligned}
0 &= \tfrac{d}{dt}\left\{ r^2\dot\varphi \sin^2\theta \right\} \\
&= 2r\dot r\dot\varphi \sin^2\theta \,+\, r^2\ddot\varphi \sin^2\theta \,+\, 2\, r^2\dot\varphi\,\dot\theta\, \sin\theta\cos\theta
\end{aligned}
$$

or

$$
2\dot r\dot\varphi \sin\theta \,+\, r\ddot\varphi \sin\theta \,+\, 2\, r\dot\varphi\,\dot\theta\, \cos\theta \;=\; 0 \tag{12}
$$

which is exactly identical to the second line of (7). The equivalence of the third equations then follows from $\frac{d}{dt}(r^2\dot\theta) = 2r\dot r\dot\theta \,+\, r^2\ddot\theta$.

Remarks:

1) Last week we derived the Euler-Lagrange equations only for the special case where a particle moves in $\mathbb{R}^3$ on a 2-dimensional surface. There the generalized coordinates were $(q_1, q_2) = (u, v)$. Here we have a situation where we have written down the Euler-Lagrange equations for 3 generalized coordinates $(q_1, q_2, q_3) = (r, \varphi, \theta)$, so this also seems to work. In fact, the formalism continues to work if one considers the motion of N particles on an f-dimensional submanifold of $\mathbb{R}^{3N}$; we will look at this again next week. One then has f generalized coordinates $q_1, \cdots, q_f$ and obtains f Euler-Lagrange equations

$$
\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot q_i} \;-\; \tfrac{\partial L}{\partial q_i} \;=\; 0\,, \qquad i = 1, \cdots, f\,. \tag{13}
$$

2) The second equation of (11) reads

$$
\tfrac{d}{dt}\left\{ r^2\dot\varphi \sin^2\theta \right\} \;=\; 0
$$

or

$$
r^2\dot\varphi \sin^2\theta \;=\; \text{constant}\,, \tag{14}
$$

so we have a conserved quantity. Conserved quantities are always obtained immediately from the Euler-Lagrange equations whenever the Lagrangian L does not depend on a particular q_i, since then $\frac{\partial L}{\partial q_i} = 0$. In our case, L was independent of φ. One then also says that the corresponding q_i, here φ, is a 'cyclic coordinate'. In the exercises we show that in this case the conserved quantity is given by the z-component of the orbital angular momentum.

Example 2: Motion of a particle under the influence of gravity in a spherical bowl (spherical pendulum)

As in the last example, we parametrize the motion by spherical coordinates,

$$
\vec{x}_t \;=\; \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \;=\; R \begin{pmatrix} \cos\varphi_t \,\sin\theta_t \\ \sin\varphi_t \,\sin\theta_t \\ \cos\theta_t \end{pmatrix} \tag{15}
$$

with

$$
\begin{aligned}
\varphi_t &\in [0, 2\pi) \\
\theta_t &\in \left[\tfrac{\pi}{2}, \pi\right] \qquad \text{only the lower half-sphere}
\end{aligned}
$$

In this case, however, the radius R is not variable, but is given by the radius of the bowl, which we have assumed to be spherical. Gravity acts, as usual, in the direction of the negative z-axis and is thus given by $\vec{F} = (0, 0, -mg) = -\nabla V(\vec{x})$ with potential

$$V(\vec{x}) \;=\; V(x,y,z) \;\;=\;\; mg\,z \;\;=\;\; mg\,R\,\cos\theta \tag{16}$$

This gives the Lagrangian

$$\begin{aligned}
L \;&=\; \text{kinetic energy} \;-\; \text{potential energy} \\
&=\; T \;-\; V \\
&=\; \tfrac{m}{2}\,\dot{\vec{x}}^{\,2} \;-\; mg\,z \\
&\overset{(9)}{=}\; \tfrac{m}{2}\Big\{\,\dot{r}^2 \;+\; r^2\dot{\varphi}^2\sin^2\theta \;+\; r^2\dot{\theta}^2\,\Big\} \;-\; mg\,R\,\cos\theta \\
&=\; \tfrac{m}{2}\Big\{\,0 \;+\; R^2\dot{\varphi}^2\sin^2\theta \;+\; R^2\dot{\theta}^2\,\Big\} \;-\; mg\,R\,\cos\theta \\
&=\; \tfrac{m}{2}\,R^2\Big\{\,\dot{\varphi}^2\sin^2\theta \;+\; \dot{\theta}^2\,\Big\} \;-\; mg\,R\,\cos\theta
\end{aligned} \tag{17}$$

The Euler-Lagrange equations are given by

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\varphi}} \;-\; \tfrac{\partial L}{\partial \varphi} \;\;=\;\; 0 \tag{18}$$

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{\theta}} \;-\; \tfrac{\partial L}{\partial \theta} \;\;=\;\; 0 \tag{19}$$

We can again divide out the mass m, and the system (18,19) is then equivalent to

$$\begin{aligned}
\tfrac{d}{dt}\big\{\,R^2\dot{\varphi}\sin^2\theta\,\big\} \;-\; 0 \;\;&=\;\; 0 \\
\tfrac{d}{dt}\big\{\,R^2\dot{\theta}\,\big\} \;-\; R^2\dot{\varphi}^2\sin\theta\cos\theta \;-\; gR\sin\theta \;\;&=\;\; 0
\end{aligned}$$

or

$$\tfrac{d}{dt}\big\{\,\dot{\varphi}\sin^2\theta\,\big\} \;\;=\;\; 0 \tag{20}$$

$$\ddot{\theta} \;-\; \dot{\varphi}^2\sin\theta\cos\theta \;-\; \tfrac{g}{R}\sin\theta \;\;=\;\; 0 \tag{21}$$

From equation (20) it follows that

$$\dot{\varphi} \;\;=\;\; \tfrac{c}{\sin^2\theta} \tag{22}$$

where c is a conserved quantity determined by the initial conditions. We can substitute (22) into (21) and obtain

$$\ddot{\theta} \;-\; \tfrac{c^2}{\sin^4\theta}\,\sin\theta\cos\theta \;-\; \tfrac{g}{R}\sin\theta \;\;=\;\; 0$$

or

$$\ddot{\theta} \;-\; c^2\,\tfrac{\cos\theta}{\sin^3\theta} \;-\; \tfrac{g}{R}\sin\theta \;\;=\;\; 0 \tag{23}$$

From equation (23) we can obtain a further conserved quantity by multiplying the equation by $\dot{\theta}$:

$$\begin{aligned}
\dot{\theta}\,\ddot{\theta} \;-\; \Big\{c^2\,\tfrac{\cos\theta}{\sin^3\theta} \;+\; \tfrac{g}{R}\sin\theta\Big\}\dot{\theta} \;\;&=\;\; 0 \\
\Leftrightarrow \qquad \tfrac{d}{dt}\Big\{\tfrac{1}{2}\,\dot{\theta}^2 \;+\; \tfrac{1}{2}\,\tfrac{c^2}{\sin^2\theta} \;+\; \tfrac{g}{R}\cos\theta\Big\} \;\;&=\;\; 0
\end{aligned}$$

Thus we get

$$\frac{1}{2}\,\dot{\theta}^2 \;+\; \frac{1}{2}\,\frac{c^2}{\sin^2\theta} \;+\; \frac{g}{R}\cos\theta \;\;=\;\; E \;\;=\;\; \text{constant} \tag{24}$$

We discuss the behaviour of the solutions of the equations of motion a little further on the new exercise sheet.