week3: The Euler-Lagrange Equations of Classical Mechanics: Derivation for 1 particle

We consider a small point mass or a small ball of mass m, which moves under the influence of gravity $F = mg$ on a two-dimensional surface. One may imagine, for instance, that the ball moves in a large bowl, where the bowl perhaps has the shape of a half sphere. Or the ball moves in a funnel (in german: Trichter) that is rotationally symmetric (with respect to the z-axis), which we can, for instance, parametrize as follows:

$$\vec{x}(r,\varphi) \;=\; \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ f(r) \end{pmatrix}$$

with $(r,\varphi) \in I$, with parameter domain, say, $I = [0, r_{\max}) \times [0, 2\pi)$. The choice

$$f(r) \;=\; c\,r$$

gives a cone-shaped funnel with the ball's rest position at $\vec{x} = (0,0,0)$, and for the choice

$$f(r) \;=\; -\sqrt{R^2 - r^2}$$

we obtain a bowl, a half sphere of radius R, with the ball's rest position at $\vec{x} = (0,0,-R)$.

In general, we thus consider the motion of a ball of mass m on a two-dimensional surface $\mathcal{F}$, for which we can, for instance, give the parametrization

$$\mathcal{F} \;=\; \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \vec{x} \;=\; \vec{x}(u,v) \;=\; \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix},\;\; (u,v) \in I \right\} \tag{1}$$

We prescribe an initial position and an initial velocity at time $t = 0$ and then want to determine the motion of the particle for arbitrary $t > 0$.

In order to take the constraint $\vec{x} \in \mathcal{F}$ into account from the outset, we make the ansatz

$$\vec{x}_t \;=\; \vec{x}\big(u(t), v(t)\big) \;=\; \vec{x}(u_t, v_t)\;. \tag{2}$$

We can then regard $\vec{x}$ either as a function of the two variables (u,v), namely as a parametrization of the surface $\mathcal{F}$, or, if we substitute concrete functions of time for u and v, we can also regard $\vec{x}_t$ simply as a function of one variable, namely the time t. By the chain rule, we have

$$\dot{\vec{x}}_t \;=\; \tfrac{\partial \vec{x}}{\partial u}\,\dot{u} \;+\; \tfrac{\partial \vec{x}}{\partial v}\,\dot{v} \tag{3}$$

$\vec{x}_t$ must satisfy Newton's equation of motion,

$$m\ddot{\vec{x}} \;=\; \vec{F} \tag{4}$$

The force $\vec{F}$ is, first of all, given by gravity

$$\vec{F}_{\text{schwer}} \;=\; -mg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5}$$

However, this is not the only force acting on the ball. Consider the example with the bowl, and suppose that the ball moves with high speed along horizontal circles near the upper border of the bowl. Centrifugal forces then act on the ball, which must be exactly compensated by the wall of the bowl. That is, the $\vec{F}$ in equation (4) consists of two components:

$$\vec{F} \;=\; \vec{F}_{\text{schwer}} \,+\, \vec{Z} \tag{6}$$

where $\vec{Z}$ is a constraint force that must exactly compensate all force components acting perpendicular to the surface $\mathcal{F}$. Note that these perpendicular force components do not come only from $\vec{F}_{\text{schwer}}$ alone but also depend on the instantaneous state of motion of the ball – think of the example with the centrifugal forces – so we have

$$\vec{Z} \;=\; \vec{Z}(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}) \;\perp\; \mathcal{F} \tag{7}$$

At every point $\vec{x}(u,v)$ of the surface $\mathcal{F}$ we have the tangent vectors $\vec{x}_u := \partial\vec{x}/\partial u$ and $\vec{x}_v := \partial\vec{x}/\partial v$, as well as the vector

$$\vec{x}_\perp \;:=\; \vec{x}_u \times \vec{x}_v \tag{8}$$

which is given by the cross product of $\vec{x}_u$ and $\vec{x}_v$ and which is perpendicular to $\mathcal{F}$. In other words, at every point $\vec{x}(u,v)$ of the surface $\mathcal{F}$ we can specify the basis

$$\left\{ \vec{x}_u \,,\, \vec{x}_v \,,\, \vec{x}_\perp \right\} \tag{9}$$

and we can expand accelerations and forces with respect to this basis. In general, of course, this basis is neither orthogonal nor orthonormal, which is why it is not entirely straightforward to compute the expansion coefficients. But they are 3 linearly independent vectors: $\vec{x}_u$ and $\vec{x}_v$ are tangent to $\mathcal{F}$, and $\vec{x}_\perp$ is perpendicular to $\mathcal{F}$. We then take Newton's equation of motion (4) with the constraint force $\vec{Z}$ from (7) which only has a $\vec{x}_\perp$-component,

$$m\ddot{\vec{x}} \;=\; \vec{F} \;=\; \vec{F}_{\text{schwer}} \,+\, \vec{Z} \;=\; \vec{F}_{\text{schwer}} \,+\, \lambda\, \vec{x}_\perp \tag{10}$$

and consider its respective $(u, v, \perp)$-components. The u and v components of that equation then will give the Euler-Lagrange equations. We remark that $\lambda = \lambda(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})$ usually depends on the current state of the particle under consideration.

To this end we assume that the force generating the motion, here it is gravity, is given by a potential:

$$\vec{F}_{\text{schwer}} \;=\; -\nabla V \;=\; -\left(\tfrac{\partial V}{\partial x_1} \,,\, \tfrac{\partial V}{\partial x_2} \,,\, \tfrac{\partial V}{\partial x_3} \right) \tag{11}$$

We obtain the gravity force (5) by choosing

$$V(\vec{x}) \;=\; V(x_1, x_2, x_3) \;\;=\;\; mg\, x_3 \tag{12}$$

Then (10) becomes

$$m\ddot{\vec{x}} \;+\; \nabla V(\vec{x}) \;\;=\;\; \vec{Z} \;\;=\;\; \lambda\, \vec{x}_\perp \tag{13}$$

We can get rid of $\vec{Z}$ by taking the scalar product with $\vec{x}_u$ and $\vec{x}_v$:

$$\begin{aligned} \big\{\, m\ddot{\vec{x}} \;+\; \nabla V(\vec{x}) \,\big\}\, \vec{x}_u \;\;&=\;\; 0 \\ \big\{\, m\ddot{\vec{x}} \;+\; \nabla V(\vec{x}) \,\big\}\, \vec{x}_v \;\;&=\;\; 0 \end{aligned} \tag{14}$$

We now have

$$\begin{aligned} \nabla V(\vec{x})\, \vec{x}_u \;\;&=\;\; \tfrac{\partial V}{\partial x_1}\, \tfrac{\partial x_1}{\partial u} \;+\; \tfrac{\partial V}{\partial x_2}\, \tfrac{\partial x_2}{\partial u} \;+\; \tfrac{\partial V}{\partial x_3}\, \tfrac{\partial x_3}{\partial u} \;\;=\;\; \tfrac{\partial V}{\partial u} \\ \nabla V(\vec{x})\, \vec{x}_v \;\;&=\;\; \tfrac{\partial V}{\partial x_1}\, \tfrac{\partial x_1}{\partial v} \;+\; \tfrac{\partial V}{\partial x_2}\, \tfrac{\partial x_2}{\partial v} \;+\; \tfrac{\partial V}{\partial x_3}\, \tfrac{\partial x_3}{\partial v} \;\;=\;\; \tfrac{\partial V}{\partial v} \end{aligned} \tag{15}$$

where on the right-hand side of (15) we regard V as a function of the variables (u, v). A mathematician would perhaps write $W(u, v) := V\big(\vec{x}(u, v)\big)$ here and then write $\frac{\partial W}{\partial u}$ on the right-hand side of (15), but we want to stick to physicist's conventions here, which of course also means that the reader must keep in mind that, for $\frac{\partial V}{\partial u}$, it is the function $V\big(\vec{x}(u, v)\big)$ which is to be differentiated, while for $\frac{\partial V}{\partial x_3}$, the function $V(x_1, x_2, x_3)$ must be differentiated.

From (14) and (15) we thus obtain the system

$$\begin{aligned} m\ddot{\vec{x}}\, \vec{x}_u \;+\; \tfrac{\partial V}{\partial u} \;\;&=\;\; 0 \\ m\ddot{\vec{x}}\, \vec{x}_v \;+\; \tfrac{\partial V}{\partial v} \;\;&=\;\; 0 \end{aligned} \tag{16}$$

Let us now consider the scalar products

$$\ddot{\vec{x}}\, \vec{x}_u \;, \quad \ddot{\vec{x}}\, \vec{x}_v$$

We have

$$\dot{\vec{x}} \;\;=\;\; \vec{x}_u\, \dot{u} \;+\; \vec{x}_v\, \dot{v} \tag{17}$$

$$\ddot{\vec{x}} \;\;=\;\; \vec{x}_{uu}\, \dot{u}^2 \;+\; 2\, \vec{x}_{uv}\, \dot{u}\dot{v} \;+\; \vec{x}_{vv}\, \dot{v}^2 \;+\; \vec{x}_u\, \ddot{u} \;+\; \vec{x}_v\, \ddot{v} \tag{18}$$

where we continue to use the abbreviations $\vec{x}_u := \partial\vec{x}/\partial u$ and $\vec{x}_{uv} := \partial^2\vec{x}/\partial u\partial v$. The notation $\vec{x}_t$, however, does not mean a time derivative, but still stands for $\vec{x}_t = \vec{x}(t) = \vec{x}\big(u(t), v(t)\big) = \vec{x}(u_t, v_t)$. Let us consider the kinetic energy

$$T \;\;=\;\; \tfrac{m}{2}\, \dot{\vec{x}}^2 \;\;\overset{(17)}{=}\;\; \tfrac{m}{2}\, \big\{ \vec{x}_u^{\,2}\, \dot{u}^2 \;+\; 2\vec{x}_u\vec{x}_v\, \dot{u}\dot{v} \;+\; \vec{x}_v^{\,2}\, \dot{v}^2 \big\} \tag{19}$$

Here $\vec{x}$ and also $\vec{x}_u$ and $\vec{x}_v$, are functions of u and v, or of u_t and v_t. In equation (19), the quantities $\dot{u} = \dot{u}_t$ and $\dot{v} = \dot{v}_t$ still appear explicitly on the right-hand side. We now want to regard T as a function of the 4 quantities $(u, v, \dot{u}, \dot{v})$,

$$T \;=\; T(u_t, v_t, \dot{u}_t, \dot{v}_t) \;\;=\;\; \tfrac{m}{2} \Big\{ \, \vec{x}_u(u_t, v_t)^2 \, \dot{u}_t^2 \;+\; 2\, \vec{x}_u(u_t, v_t) \, \vec{x}_v(u_t, v_t) \, \dot{u}_t \dot{v}_t \;+\; \vec{x}_v(u_t, v_t)^2 \, \dot{v}_t^2 \, \Big\}$$

or, if we drop the time index t,

$$T \;=\; T(u, v, \dot{u}, \dot{v}) \;\;=\;\; \tfrac{m}{2} \Big\{ \, \vec{x}_u(u, v)^2 \, \dot{u}^2 \;+\; 2\, \vec{x}_u(u, v) \, \vec{x}_v(u, v) \, \dot{u}\dot{v} \;+\; \vec{x}_v(u, v)^2 \, \dot{v}^2 \, \Big\} \tag{20}$$

and, in the next theorem, we also want to consider the $(u, v, \dot{u}, \dot{v})$ as 4 independent quantities. Then the following statement holds:

Theorem 2.1.1: Let $\vec{x} = \vec{x}(u, v)$ be a parametrization of $\mathcal{F}$, $\vec{x}_t = \vec{x}(u_t, v_t)$, and let the function $T = T(u, v, \dot{u}, \dot{v})$ be given by (20). Then

$$\begin{aligned} \tfrac{d}{dt} \tfrac{\partial T}{\partial \dot{u}} \;-\; \tfrac{\partial T}{\partial u} \;\;&=\;\; m \ddot{\vec{x}} \, \vec{x}_u \\ \tfrac{d}{dt} \tfrac{\partial T}{\partial \dot{v}} \;-\; \tfrac{\partial T}{\partial v} \;\;&=\;\; m \ddot{\vec{x}} \, \vec{x}_v \end{aligned} \tag{21}$$

Proof: By (19) we have

$$\tilde{T} \;:=\; T/m \;\;=\;\; \tfrac{1}{2} \left\{ \vec{x}_u^{\,2} \, \dot{u}^2 \;+\; 2 \vec{x}_u \vec{x}_v \, \dot{u}\dot{v} \;+\; \vec{x}_v^{\,2} \, \dot{v}^2 \right\} \tag{22}$$

Hence

$$\begin{aligned} \tfrac{\partial \tilde{T}}{\partial \dot{u}} \;\;&=\;\; \vec{x}_u^{\,2} \, \dot{u} \;+\; \vec{x}_u \vec{x}_v \, \dot{v} \;\;=\;\; \vec{x}_u \left(\vec{x}_u \, \dot{u} \;+\; \vec{x}_v \, \dot{v} \right) \;\;=\;\; \vec{x}_u \, \dot{\vec{x}} \\ \tfrac{d}{dt} \tfrac{\partial \tilde{T}}{\partial \dot{u}} \;\;&=\;\; \tfrac{d}{dt} \vec{x}_u \, \dot{\vec{x}} \;+\; \vec{x}_u \, \ddot{\vec{x}} \end{aligned}$$

and, using $\vec{x}_{uv} = \vec{x}_{vu}$ in the last line,

$$\begin{aligned} \tfrac{\partial \tilde{T}}{\partial u} \;\;&=\;\; \vec{x}_{uu} \vec{x}_u \, \dot{u}^2 \;+\; (\vec{x}_{uu} \vec{x}_v + \vec{x}_u \vec{x}_{uv}) \, \dot{u}\dot{v} \;+\; \vec{x}_v \vec{x}_{uv} \, \dot{v}^2 \\ &=\;\; (\vec{x}_{uu} \dot{u} \;+\; \vec{x}_{uv} \dot{v}) \, (\vec{x}_u \dot{u} \;+\; \vec{x}_v \dot{v}) \\ &=\;\; \tfrac{d}{dt} \vec{x}_u \, \dot{\vec{x}} \end{aligned}$$

Thus,

$$\tfrac{d}{dt} \tfrac{\partial \tilde{T}}{\partial \dot{u}} \;-\; \tfrac{\partial \tilde{T}}{\partial u} \;\;=\;\; \tfrac{d}{dt} \vec{x}_u \, \dot{\vec{x}} \;+\; \vec{x}_u \, \ddot{\vec{x}} \;-\; \tfrac{d}{dt} \vec{x}_u \, \dot{\vec{x}} \;\;=\;\; \vec{x}_u \, \ddot{\vec{x}}$$

The equation with the v-derivatives follows analogously. ■

We can thus write the system (16) as

$$\begin{aligned} \tfrac{d}{dt} \tfrac{\partial T}{\partial \dot{u}} \;-\; \tfrac{\partial T}{\partial u} \;+\; \tfrac{\partial V}{\partial u} \;\;&=\;\; 0 \\ \tfrac{d}{dt} \tfrac{\partial T}{\partial \dot{v}} \;-\; \tfrac{\partial T}{\partial v} \;+\; \tfrac{\partial V}{\partial v} \;\;&=\;\; 0 \end{aligned}$$

or

$$\begin{aligned} \tfrac{d}{dt}\tfrac{\partial T}{\partial \dot{u}} - \tfrac{\partial (T-V)}{\partial u} &= 0 \\ \tfrac{d}{dt}\tfrac{\partial T}{\partial \dot{v}} - \tfrac{\partial (T-V)}{\partial v} &= 0 \end{aligned}$$

Since, by assumption, V depends only on $\vec{x}$ but not on $\dot{\vec{x}}$, we have $\frac{\partial V}{\partial \dot{u}} = 0 = \frac{\partial V}{\partial \dot{v}}$, and we can finally write

$$\begin{aligned} \tfrac{d}{dt}\tfrac{\partial (T-V)}{\partial \dot{u}} - \tfrac{\partial (T-V)}{\partial u} &= 0 \\ \tfrac{d}{dt}\tfrac{\partial (T-V)}{\partial \dot{v}} - \tfrac{\partial (T-V)}{\partial v} &= 0 \end{aligned} \tag{23}$$

We summarize our results in the following theorem.

Theorem 2.1.2: Let a particle of mass m move under the influence of a potential $V(\vec{x})$ in $\mathbb{R}^3$. Through this potential the force

$$\vec{F} = -\nabla V(\vec{x})$$

acts on the particle. Furthermore, suppose the particle is subject to mechanical constraints such that its motion can take place only within a surface

$$\mathcal{F} = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \vec{x} = \vec{x}(u,v) = \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix}, \; (u,v) \in I \right\}$$

Then:

a) The trajectory $\vec{x}_t = \vec{x}(u_t, v_t)$ satisfies the equation of motion

$$m\ddot{\vec{x}} + \nabla V(\vec{x}) = \lambda \vec{x}_\perp$$

with $\vec{x}_\perp = \vec{x}_u \times \vec{x}_v$. Here $\lambda = \lambda(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})$ is a function with whose help the constraint force $\vec{Z} = \lambda \vec{x}_\perp$ caused by the mechanical constraints can be computed. The constraint force $\vec{Z}$ is a real force, not merely a mathematical device.

b) If we define the Lagrangian L by

$$L := T - V = \tfrac{m}{2}\dot{\vec{x}}^2 - V(\vec{x}) \tag{24}$$

then the trajectory $\vec{x}_t = \vec{x}(u_t, v_t)$ of the particle can be obtained from the Euler-Lagrange equations

$$\begin{aligned} \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{u}} - \tfrac{\partial L}{\partial u} &= 0 \\ \tfrac{d}{dt}\tfrac{\partial L}{\partial \dot{v}} - \tfrac{\partial L}{\partial v} &= 0 \end{aligned} \tag{25}$$

Remarks:

1) To set up and solve the Euler-Lagrange equations (25), that is to compute the trajectory, knowledge of the constraint force $\vec{Z} = \lambda \vec{x}_\perp$ is not required. The relevant part of the theorem is thus typically part (b); part (a) is 'just for information', so to speak, just for a better understanding.

2) The parameters (u, v) (or any choice of such parameters) that ensure the constraint

$$\vec{x} \;=\; \vec{x}(u,v) \quad \in \quad \mathcal{F}$$

are called generalized coordinates for the given mechanical problem. Typically they are denoted in physics textbooks by $q_1, q_2, \cdots, q_f$, with f = number of degrees of freedom (so here $f = 2$).