week2: Solutions to Exercise Sheet 1:
Undamped, Damped, and Forced Harmonic Oscillator

First of all, the following general facts hold:

(i) A homogeneous ('there is a zero on the right-hand side') linear second-order differential equation (ODE, ordinary differential equation) has two linearly independent solutions, that is, the general solution contains two constants.

(ii) The general solution of an inhomogeneous linear second-order ODE is given by the general solution of the homogeneous equation plus a special (or 'particular') solution of the inhomogeneous equation.

With this we can now solve the exercise:

a) Obviously $\sin\omega t$ and $\cos\omega t$ are solutions of

$$\ddot{x}_t \;+\; \omega^2 x_t \;=\; 0 \tag{1}$$

Hence the general solution of (1) reads:

$$x(t) \;=\; a\,\cos\omega t \;+\; b\,\sin\omega t$$

With the initial conditions

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

we obtain

$$x(t) \;=\; x_0\,\cos\omega t \;+\; \tfrac{\dot{x}_0}{\omega}\,\sin\omega t \tag{2}$$

as the unique solution of (1).

b) We make the complex ansatz

$$x(t) \;=\; e^{i\lambda t}$$

with a complex $\lambda \in \mathbb{C}$ and obtain

$$(i\lambda)^2 \;+\; i\lambda\gamma \;+\; \omega^2 \;=\; 0$$

or, with $\gamma = 2\mu$,

$$\begin{aligned} & -\lambda^2 \;+\; 2i\lambda\mu \;+\; \omega^2 \;=\; 0 \\ \Leftrightarrow \qquad & \lambda^2 \;-\; 2i\lambda\mu \;-\; \omega^2 \;=\; 0 \end{aligned}$$

The solution of this quadratic equation is given by

$$\begin{aligned} \lambda_\pm &= i\mu \pm \sqrt{(i\mu)^2 + \omega^2} \\ &= i\mu \pm \sqrt{\omega^2 - \mu^2} \end{aligned} \tag{3}$$

We consider the cases $\mu < \omega$, $\mu = \omega$ and $\mu > \omega$.

Case 1: $\mu < \omega$, weak damping: Equation (3) has the solutions

$$\lambda_\pm = \pm\,\omega_\mu + i\mu \tag{4}$$

with a real frequency

$$\omega_\mu := \sqrt{\omega^2 - \mu^2} \in \mathbb{R}$$

This gives us the complex solutions

$$x_\pm(t) = e^{i\lambda_\pm t} = e^{-\mu t}\, e^{\pm i\omega_\mu t} \tag{5}$$

Because of

$$\begin{aligned} & & \ddot{x}_t + \gamma\,\dot{x}_t + \omega^2 x_t &= 0 \qquad (6) \\ \Rightarrow & & \mathrm{Re}\big[\,\ddot{x}_t + \gamma\,\dot{x}_t + \omega^2 x_t\,\big] &= 0 \\ \Leftrightarrow & & \tfrac{d^2}{dt^2}\mathrm{Re}[x_t] + \gamma\,\tfrac{d}{dt}\mathrm{Re}[x_t] + \omega^2\mathrm{Re}[x_t] &= 0 \end{aligned}$$

the real part and imaginary part of the solution (5) are also solutions of the ODE (6). Hence the general real solution of (6) is given by

$$x(t) = e^{-\mu t}\big[\,a\,\cos\omega_\mu t + b\,\sin\omega_\mu t\,\big] \tag{7}$$

We have

$$\dot{x}(t) = -\mu\, e^{-\mu t}\big[\,a\,\cos\omega_\mu t + b\,\sin\omega_\mu t\,\big] + e^{-\mu t}\big[\,-a\omega_\mu\,\sin\omega_\mu t + b\omega_\mu\,\cos\omega_\mu t\,\big]$$

and therefore

$$\begin{aligned} x(0) &= a \overset{!}{=} x_0 \\ \dot{x}(0) &= -\mu\, a + b\,\omega_\mu \overset{!}{=} \dot{x}_0 \end{aligned}$$

Thus,

$$\begin{aligned} a &= x_0 \\ b &= \tfrac{\dot{x}_0 + \mu x_0}{\omega_\mu} \end{aligned}$$

and hence

$$x(t) = e^{-\mu t}\Big[\,x_0\,\cos\omega_\mu t + \tfrac{\dot{x}_0 + \mu x_0}{\omega_\mu}\,\sin\omega_\mu t\,\Big] \tag{8}$$

Case 2: $\mu > \omega$, strong damping: Equation (3) has the solutions

$$\lambda_\pm \;=\; i\mu \,\pm\, i\sqrt{\mu^2-\omega^2} \;=:\; i\,\kappa_\pm \tag{9}$$

with the real and positive decay rates

$$\kappa_\pm \;:=\; \mu \,\pm\, \sqrt{\mu^2-\omega^2} \;>\; 0$$

This gives us the general real solution

$$x(t) \;=\; a_+\, e^{-\kappa_+ t} \,+\, a_-\, e^{-\kappa_- t} \tag{10}$$

Because of

$$\begin{aligned} x(0) \;&=\; a_+ + a_- \;\stackrel{!}{=}\; x_0 \\ \dot{x}(0) \;&=\; -\kappa_+ a_+ - \kappa_- a_- \;\stackrel{!}{=}\; \dot{x}_0 \end{aligned}$$

we obtain, by multiplying the first equation by κ_+ or κ_- and adding it to the second one,

$$\begin{aligned} \kappa_+ a_- \,-\, \kappa_- a_- \;&=\; \kappa_+ x_0 + \dot{x}_0 \\ \kappa_- a_+ \,-\, \kappa_+ a_+ \;&=\; \kappa_- x_0 + \dot{x}_0 \end{aligned}$$

and therefore

$$\begin{aligned} a_+ \;&=\; -\,\tfrac{\kappa_- x_0 + \dot{x}_0}{\kappa_+ - \kappa_-} \\ a_- \;&=\; +\,\tfrac{\kappa_+ x_0 + \dot{x}_0}{\kappa_+ - \kappa_-} \end{aligned}$$

We can then substitute this into (10) and obtain the solution for the given initial conditions.

Case 3: $\mu = \omega$, critically damped case: Equation (3) has only one solution,

$$\lambda \;=\; i\,\mu \,\pm\, 0 \;=\; i\,\mu$$

and for the time being we only have one solution

$$x(t) \;=\; e^{-\mu t}$$

In such situations,

$$x(t) \;=\; t\, e^{-\mu t}$$

gives a further solution; let us quickly verify this: we have

$$\begin{aligned} \dot{x}(t) \;&=\; e^{-\mu t} \,-\, \mu\, t\, e^{-\mu t} \;=\; (1-\mu t)\, e^{-\mu t} \\ \ddot{x}(t) \;&=\; (-\mu)\, e^{-\mu t} \,-\, \mu(1-\mu t)\, e^{-\mu t} \;=\; (\mu^2 t \,-\, 2\mu)\, e^{-\mu t} \end{aligned}$$

and therefore obtain

$$\begin{aligned} \ddot{x}_t \,+\, \gamma\, \dot{x}_t \,+\, \omega^2 x_t \;&=\; \ddot{x}_t \,+\, 2\mu\, \dot{x}_t \,+\, \mu^2 x_t \\ &=\; \Big\{ (\mu^2 t \,-\, 2\mu) \,+\, 2\mu(1-\mu t) \,+\, \mu^2 t \Big\} e^{-\mu t} \\ &=\; 0 \end{aligned}$$

The general solution thus reads

$$x(t) \;=\; (a+bt)\,e^{-\mu t}$$

with

$$\dot{x}(t) \;=\; \big[\, b \,-\, \mu(a+bt)\,\big]\,e^{-\mu t}$$

The initial conditions then yield

$$a \;=\; x_0$$

and

$$\begin{aligned} b-\mu a \;=\; b-\mu x_0 \;&=\; \dot{x}_0 \\ b \;&=\; \dot{x}_0+\mu x_0 \end{aligned}$$

so

$$x(t) \;=\; \big[\, x_0 \,+\, (\dot{x}_0+\mu x_0)t\,\big]\,e^{-\mu t} \tag{11}$$

c) We now have an inhomogeneous ODE,

$$\ddot{x}_t \;+\; \gamma\,\dot{x}_t \;+\; \omega^2\,x_t \;=\; a_0\,\sin(\omega_0 t) \tag{12}$$

and have to find a special solution of the inhomogeneous equation. The general solution of the homogeneous equation

$$\ddot{x}_t \;+\; \gamma\,\dot{x}_t \;+\; \omega^2\,x_t \;=\; 0$$

we already determined in part (b). We consider the complex equation

$$\ddot{z}_t \;+\; \gamma\,\dot{z}_t \;+\; \omega^2\,z_t \;=\; a_0\,e^{i\omega_0 t} \tag{13}$$

If we have a $z(t)$ which satisfies (13), then, with a real a_0, we obtain

$$\begin{aligned} &\mathrm{Im}\big[\,\ddot{z}_t \;+\; \gamma\,\dot{z}_t \;+\; \omega^2\,z_t\,\big] \;&=\;& \mathrm{Im}\big[\,a_0\,e^{i\omega_0 t}\,\big] \\ \Leftrightarrow\quad &\tfrac{d^2}{dt^2}\mathrm{Im}[z_t] \;+\; \gamma\,\tfrac{d}{dt}\mathrm{Im}[z_t] \;+\; \omega^2\mathrm{Im}[z_t] \;&=\;& a_0\,\sin(\omega_0 t) \end{aligned}$$

so then

$$x(t) \;:=\; \mathrm{Im}[z_t]$$

is a solution of (12). To find a solution of (13), we make the ansatz

$$z(t) \;=\; a\,e^{i\omega_0 t}$$

and obtain

$$a\big\{\,-\omega_0^2 \;+\; i\omega_0\gamma \;+\; \omega^2\,\big\}\,e^{i\omega_0 t} \;=\; a_0\,e^{i\omega_0 t}$$

or

$$a \;=\; \frac{a_0}{\omega^2-\omega_0^2 \,+\, i\gamma\omega_0} \;=\; a_0\,\frac{\omega^2-\omega_0^2 \,-\, i\gamma\omega_0}{(\omega^2-\omega_0^2)^2 \,+\, (\gamma\omega_0)^2}$$

Thus we obtain the complex solution

$$z(t) \;=\; a_0\,\frac{\omega^2-\omega_0^2 \,-\, i\gamma\omega_0}{(\omega^2-\omega_0^2)^2 \,+\, (\gamma\omega_0)^2}\,\left[\,\cos(\omega_0 t) \,+\, i\,\sin(\omega_0 t)\,\right]$$

and have to determine its imaginary part: a special solution of the inhomogeneous real ODE (12) is thus given by

$$\begin{aligned} x_{\rm inhom}(t) \;&=\; {\rm Im}[z_t] \\ &=\; \frac{a_0}{(\omega^2-\omega_0^2)^2 \,+\, (\gamma\omega_0)^2}\,\left\{\,(\omega^2-\omega_0^2)\sin(\omega_0 t) \;-\; \gamma\omega_0\,\cos(\omega_0 t)\,\right\} \end{aligned} \tag{14}$$

and the general solution is then given by

$$x(t) \;=\; x_{\rm gen,hom}(t) \,+\, x_{\rm spec,inhom}(t) \tag{15}$$

where $x_{\rm gen,hom}(t)$ is the general solution of the homogeneous equation, which contains two constants, say c_1 and c_2; $x_{\rm spec,inhom}(t) = x_{\rm inhom}(t)$ is the special solution of the inhomogeneous equation from (14); and the two constants c_1 and c_2 then again have to be determined from the initial conditions

$$\begin{aligned} x(0) \;&\stackrel{!}{=}\; x_0 \\ \dot{x}(0) \;&\stackrel{!}{=}\; \dot{x}_0 \end{aligned}$$

At this point we only want to do this for the case without damping, i.e. only for $\gamma = 0$. In that case we have

$$x_{\rm spec,inhom}(t) \;=\; \frac{a_0}{\omega^2-\omega_0^2}\,\sin(\omega_0 t)$$

$$x_{\rm gen,hom}(t) \;=\; c_1\,\cos\omega t \,+\, c_2\,\sin\omega t$$

so

$$x(t) \;=\; c_1\,\cos\omega t \,+\, c_2\,\sin\omega t \,+\, \frac{a_0}{\omega^2-\omega_0^2}\,\sin(\omega_0 t) \tag{16}$$

so that

$$x(0) \;=\; c_1 \;=\; x_0$$

and

$$\dot{x}(0) \;=\; c_2\,\omega \,+\, \frac{a_0\omega_0}{\omega^2-\omega_0^2} \;\stackrel{!}{=}\; \dot{x}_0$$

Thus,

$$c_2 \;=\; \frac{\dot{x}_0}{\omega} \;-\; a_0\,\frac{\omega_0}{\omega}\,\frac{1}{\omega^2-\omega_0^2}$$

and finally we obtain

$$x(t) \;=\; x_0\,\cos\omega t \;+\; \frac{\dot{x}_0}{\omega}\,\sin\omega t \;+\; \frac{a_0}{\omega^2-\omega_0^2}\,\left\{\,-\,\frac{\omega_0}{\omega}\,\sin(\omega t) \,+\, \sin(\omega_0 t)\,\right\} \tag{17}$$

for the case $\gamma = 0$.