

week1: Brief Reminder Newtonian Mechanics

The fundamental equation of Newtonian mechanics reads:

$$\text{force} = \text{mass} \times \text{acceleration}$$

or in symbols

$$\vec{F} = m \vec{a} = m \ddot{\vec{x}} = m \frac{d^2 \vec{x}}{dt^2}$$

where $\vec{x} = \vec{x}(t)$ is the trajectory of the point particle, its position as a function of time. In many situations the force is generated by a potential. This means that the force $\vec{F} = \vec{F}(\vec{x})$ can be written as

$$\vec{F} = -\nabla V(\vec{x}) = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

One then also says that the force field $\vec{F}$ is conservative. We thus have

$$m \ddot{\vec{x}} + \nabla V(\vec{x}) = 0$$

We multiply this equation by $\dot{\vec{x}}$,

$$m \ddot{\vec{x}} \cdot \dot{\vec{x}} + \nabla V(\vec{x}) \cdot \dot{\vec{x}} = 0$$

where the dot, $\cdot$, means the scalar product in $\mathbb{R}^3$,

$$\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$$

Then, by the chain rule, with $\vec{x}(t) = (x_t, y_t, z_t)$

$$\frac{d}{dt} V(\vec{x}(t)) = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} + \frac{\partial V}{\partial z} \frac{dz}{dt} = \nabla V \cdot \dot{\vec{x}}$$

Thus we can write

$$\frac{m}{2} \frac{d}{dt} \dot{\vec{x}}^2 + \frac{d}{dt} V(\vec{x}) = 0$$

or

$$\frac{d}{dt} \left\{ \frac{m}{2} \dot{\vec{x}}^2 + V(\vec{x}) \right\} = 0$$

This means that the total energy of the system,

$$E := \frac{m}{2} \dot{\vec{x}}^2 + V(\vec{x})$$

must be constant in time. Besides the total energy E there can be further conserved quantities. If, for instance, the potential is given by a central potential, i.e. V can be written as

$$V = V(\|\vec{x}\|) = V(r)$$

then the angular momentum $\vec{L}$ (we write $\vec{p} = m \dot{\vec{x}}$ for the linear momentum)

$$\vec{L} := \vec{x} \times \vec{p} = \vec{x} \times (m \dot{\vec{x}}) = m \vec{x} \times \dot{\vec{x}}$$

is a constant of motion, since (here $\times$ denotes the vector cross product):

$$\frac{d\vec{L}}{dt} = m \underbrace{\dot{\vec{x}} \times \dot{\vec{x}}}_{=0} + m \vec{x} \times \ddot{\vec{x}} = \vec{x} \times \vec{F} = -\vec{x} \times \nabla V$$

and for a central potential $V = V(r)$ we now have, with $r = \|\vec{x}\| = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial V}{\partial x} = V'(r) \frac{\partial r}{\partial x} = V'(r) \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = V'(r) \frac{x}{r}$$

and in a similar way

$$\begin{aligned} \frac{\partial V}{\partial y} &= V'(r) \frac{y}{r} \\ \frac{\partial V}{\partial z} &= V'(r) \frac{z}{r} \end{aligned}$$

Or, as a vector equation, we can write

$$\nabla V = V'(r) \frac{\vec{x}}{r}$$

Thus, the time derivative of the angular momentum $\vec{L}$ is given by

$$\frac{d\vec{L}}{dt} = -\vec{x} \times \nabla V = -\frac{V'(r)}{r} \vec{x} \times \vec{x} = \vec{0}$$

such that $\vec{L}$ is a conserved quantity,

$$\vec{L} = \text{const}$$

for force fields which are generated by some central potential $V = V(r)$.

Perhaps the most important force fields and potentials are the following:

- 1) The $1/r$ potential, also called the Coulomb potential. It describes both the electrostatic force between two point charges as well as the gravitational force between two point masses. If a point particle with charge Q is located at the origin $\vec{x} = \vec{0}$ and another

with charge q is located at position $\vec{x}$, then the force $\vec{F}$ experienced by the charge q due to the charge Q is given by (in MKSA units)

$$\vec{F}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \frac{\vec{x}}{r}$$

with

$$\epsilon_0 = 8.854 \times 10^{-12} \frac{\text{C}}{\text{Vm}} = 8.854 \times 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$$

This force is generated by the Coulomb potential

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{qQ}{r}$$

Namely, using the formula

$$\nabla V = V'(r) \frac{\vec{x}}{r}$$

again, we obtain

$$-\nabla V = -\frac{d}{dr} \left\{ \frac{1}{4\pi\epsilon_0} \frac{qQ}{r} \right\} \frac{\vec{x}}{r} = -\left\{ -\frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \right\} \frac{\vec{x}}{r} = +\frac{1}{4\pi\epsilon_0} \frac{qQ}{r^2} \frac{\vec{x}}{r}.$$

The gravitational force is also described by a $1/r$ potential: If a point mass M is located at the origin $\vec{x} = \vec{0}$ and another point mass m is located at position $\vec{x}$, then the force $\vec{F}$ experienced by the mass m due to the point mass M is given by

$$\vec{F}(\vec{x}) = -G \frac{mM}{r^2} \frac{\vec{x}}{r}$$

with the gravitational constant

$$G = 6.674 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$$

Thus, this force is generated by the potential

$$V(r) = -G \frac{mM}{r}$$

2) The harmonic oscillator potential

$$V(\vec{x}) = c\vec{x}^2$$

In one dimension this describes, for instance, the motion of a mass on a spring: we have the linear force

$$F = -Dx$$

with spring constant D , which is generated by the one-dimensional potential

$$V(x) = \frac{D}{2} x^2$$

- 3) On point masses m located only a few meters above the Earth's surface, there acts a constant, position-independent gravitational force of the form

$$\vec{F} = -mg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

with the gravitational acceleration

$$g \approx 9.81 \frac{\text{m}}{\text{s}^2}$$

This is generated by a linear potential

$$V(x, y, z) = mgz$$

In german, this force is referred to as 'Schwerkraft' whereas the term 'Gravitationskraft' refers to the force mentioned under item (1). In english, there seems to be no extra word like 'Schwerkraft', but in both cases you just say 'gravity'.

Since in the further course of these lectures we will frequently consider the motion of a point mass under the influence of gravity, let us derive here the 'Schwerkraft' $F = mg$ from the more fundamental law of gravitation as given under item (1). To do so, let's consider the following exercise:

Exercise: We want to derive the 'Schwerkraft' $F = mg$, with $g = 9.81 \frac{\text{m}}{\text{s}^2}$ from Newton's law of gravitation

$$\vec{F}_{21} = -Gm_1m_2 \frac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|^3} \quad (1)$$

Here $\vec{F}_{21}$ is the force exerted by the point mass m_1 on the point mass m_2 , and G is the gravitational constant as given above. We consider the case with $m_1 = M$ = mass of the Earth and $m_2 = m$ is a test mass which, let's say, has been put on a table in a laboratory. We can no longer regard $m_1 = M$ as a point mass, and therefore write

$$M = \int_{\text{Earth}} dM(\vec{x}_1)$$

with small mass elements $dM(\vec{x}_1)$, which we then want to regard as point-like. By equation (1), such a mass element dM located at position $\vec{x}_1$ then exerts the force

$$d\vec{F} = d\vec{F}_{21} = -G dm_1 m_2 \frac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|^3} = -G dM(\vec{x}_1) m \frac{\vec{x} - \vec{x}_1}{\|\vec{x} - \vec{x}_1\|^3}$$

on the test mass $m = m_2$, when m is located at position $\vec{x} := \vec{x}_2$. The total force acting on m is then given by

$$\vec{F}(\vec{x}) = \int_{\text{Earth}} d\vec{F} = -Gm \int_{\text{Earth}} dM(\vec{x}_1) \frac{\vec{x} - \vec{x}_1}{\|\vec{x} - \vec{x}_1\|^3} = -Gm\rho \int_{\text{Earth}} d^3x_1 \frac{\vec{x} - \vec{x}_1}{\|\vec{x} - \vec{x}_1\|^3} \quad (2)$$

where

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{dM}{dV} = \frac{dM}{d^3x_1}$$

denotes the density of the Earth which we consider as a constant. We assume that the Earth is a sphere of radius

$$R = 6371 \text{ km}$$

Then the force $\vec{F}$ in (2) depends only on $r := \|\vec{x}\|$, that is, $\vec{F} = \vec{F}(r)$, and we obtain the following formulae:

a) Let $r \geq R$ and let $F(r) = |\vec{F}(r)|$. Then

$$F(r) = G m \rho \frac{4\pi}{3} R^3 \times \frac{1}{r^2} = G m \frac{M}{r^2} \quad (3)$$

Thus, for $r \geq R$, we could still consider the Earth as a point mass M placed at the origin $\vec{x}_1 = \vec{0}$ and then application of equation (1) would give the same result.

b) Let the test mass $m = m_2$ now be located at some height h above the Earth's surface, let's say $h = 1\text{m}$. We thus have the case $r = R + h$ with $h \ll R$. Then, since $h/R \approx 10^{-6}$,

$$F(r) \approx G m \frac{M}{R^2} \left(1 - 2\frac{h}{R}\right) \approx G m \frac{M}{R^2} = g m \quad (4)$$

with

$$g = G \frac{M}{R^2} \approx 9.81 \frac{\text{m}}{\text{s}^2}.$$

The mass of the Earth is

$$M = 5.9722 \times 10^{24} \text{ kg}.$$

Solution: We calculate the integral in (2) in the 'Advanced Mathematics' course. ■