Hochschule RheinMain
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5. Homework Assignment Advanced Classical Mechanics

Exercise 1: We consider the motion of a small ball of mass m on a cone K under the influence of gravity $\vec{F} = (0, 0, -mg)$. Let the cone have the parametrization

$$K \;=\; \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \vec{x} \,=\, \vec{x}(r,\varphi) \;=\; \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ c\,r \end{pmatrix},\;\; (r,\varphi) \in [0,R] \times [0,2\pi) \right\}$$

where c is a positive constant.

a) Choose (r, φ) as generalized coordinates and write down the Lagrange function.

b) Derive the equations of motion, the Euler-Lagrange equations, for r and φ.

This is a typical exam question.

Exercise 2: We consider the double pendulum as specified in Example 2 in `week5.pdf`. Have another look in particular at the figure on page 3. Then choose φ_1 and φ_2 as the generalized coordinates.

a) Give the constraint forces $\vec{Z}_1$ and $\vec{Z}_2$ as functions of (x_1, y_1, x_2, y_2) and the parameters $\lambda_1 = \lambda_1(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})$ and $\lambda_2 = \lambda_2(\vec{x}, \dot{\vec{x}}, \ddot{\vec{x}})$. To do so, use formulas (15) and (16) from `week5.pdf`.

b) Give the explicit formulas

$$\vec{x}_1 \;=\; \begin{pmatrix} x_1(\varphi_1,\varphi_2) \\ y_1(\varphi_1,\varphi_2) \end{pmatrix}, \qquad \vec{x}_2 \;=\; \begin{pmatrix} x_2(\varphi_1,\varphi_2) \\ y_2(\varphi_1,\varphi_2) \end{pmatrix}$$

for the parametrization by the generalized coordinates (φ_1, φ_2) and compute the quantities

$$\tfrac{\partial \vec{x}_1}{\partial \varphi_1}, \quad \tfrac{\partial \vec{x}_2}{\partial \varphi_1}, \quad \tfrac{\partial \vec{x}_1}{\partial \varphi_2}, \quad \tfrac{\partial \vec{x}_2}{\partial \varphi_2}.$$

Then express the result for these derivatives again in terms of the (x_i, y_i), not the φ_i; this makes part (c) a bit clearer.

c) Now show: with $\vec{Z} = (\vec{Z}_1, \vec{Z}_2)$ and $\vec{x} = (\vec{x}_1, \vec{x}_2)$ we have

$$\begin{aligned} \vec{Z} \cdot \tfrac{\partial \vec{x}}{\partial \varphi_1} \;&=\; \vec{Z}_1 \cdot \tfrac{\partial \vec{x}_1}{\partial \varphi_1} \;+\; \vec{Z}_2 \cdot \tfrac{\partial \vec{x}_2}{\partial \varphi_1} \;=\; 0 \\ \vec{Z} \cdot \tfrac{\partial \vec{x}}{\partial \varphi_2} \;&=\; \vec{Z}_1 \cdot \tfrac{\partial \vec{x}_1}{\partial \varphi_2} \;+\; \vec{Z}_2 \cdot \tfrac{\partial \vec{x}_2}{\partial \varphi_2} \;=\; 0 \end{aligned}$$

but the individual scalar products $\vec{Z}_i \cdot \frac{\partial \vec{x}_i}{\partial \varphi_j}$ need not necessarily be zero. We used that property in the derivation of the Euler-Lagrange equations for N particles, so it makes sense the check this for a particular case. However, this exercise is not relevant for the final exam, there won't be any constraint forces $\vec{Z}$ or $\vec{Z}_i$ in the final exam.