Hochschule RheinMain
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4. Homework Assignment Advanced Classical Mechanics

Exercise 1: We parametrize the motion of a particle of mass m in $\mathbb{R}^3$ by spherical coordinates,

$$\vec{x} \;=\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;=\; r \begin{pmatrix} \cos\varphi\, \sin\theta \\ \sin\varphi\, \sin\theta \\ \cos\theta \end{pmatrix}$$

with $r \in [0, +\infty)$, $\varphi \in [0, 2\pi)$ and $\theta \in [0, \pi]$. We introduce the following basis vectors:

$$\vec{e}_r \;:=\; \begin{pmatrix} \cos\varphi\, \sin\theta \\ \sin\varphi\, \sin\theta \\ \cos\theta \end{pmatrix}, \quad \vec{e}_\varphi \;:=\; \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \quad \vec{e}_\theta \;:=\; \begin{pmatrix} \cos\varphi\, \cos\theta \\ \sin\varphi\, \cos\theta \\ -\sin\theta \end{pmatrix}$$

The orbital angular momentum is defined by the vector

$$\vec{L} \;:=\; \vec{x} \times \vec{p} \;=\; m\, \vec{x} \times \dot{\vec{x}}$$

Show:

a) For a force-free motion $m\ddot{\vec{x}} = 0$, as we considered it in Example 1 of `week4.pdf` , we have

$$\tfrac{d}{dt}\, \vec{L} \;=\; \vec{0}\,,$$

that is, $\vec{L}$ is a conserved quantity.

b) For a motion under the influence of gravity $\vec{F} = (0, 0, -mg)$ and a constraint force of the form $\vec{Z} = \lambda\, \vec{e}_r$, we no longer have $\frac{d}{dt}\, \vec{L} = \vec{0}$, but we still have

$$\tfrac{d}{dt}\, L_3 \;=\; 0$$

where L_3 is the z-component of the angular momentum $\vec{L} = (L_1, L_2, L_3)$.

c) In spherical coordinates we have

$$\vec{L} \;=\; m\, r^2 \left\{ \dot{\theta}\, \vec{e}_\varphi \;-\; \dot{\varphi}\, \sin\theta\, \vec{e}_\theta \right\}$$

and

$$L_3 \;=\; m\, r^2\, \dot{\varphi}\, \sin^2\theta\,.$$

Exercise 2: We consider the motion of a small ball of mass m under the influence of gravity in a spherical bowl as we discussed in Example 2 of `week4.pdf` . There we parametrized the motion by spherical coordinates,

$$\vec{x}_t \;=\; \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} \;=\; R \begin{pmatrix} \cos\varphi_t \, \sin\theta_t \\ \sin\varphi_t \, \sin\theta_t \\ \cos\theta_t \end{pmatrix}$$

where R now is the constant radius of the bowl. Using the Lagrange formalism, we derived the following equations of motion:

$$\tfrac{d}{dt}\big\{\, \dot{\varphi} \sin^2\theta \,\big\} \;=\; 0 \tag{1}$$

$$\ddot{\theta} \;-\; \dot{\varphi}^2 \sin\theta\cos\theta \;-\; \tfrac{g}{R}\sin\theta \;=\; 0 \tag{2}$$

a) How must the initial conditions be chosen if the ball is to move on horizontal circles? Try to characterize all such possible initial conditions.

b) For $\dot{\varphi} = 0$ we may consider the motion as that of a planar simple pendulum of length R. The rest position of the pendulum is at $\vec{x} = (0, 0, -R)$, i.e. $\theta = \pi$. Substitute $\theta = \pi - \alpha$ and consider the case of small displacements $\alpha \ll 1$, so that the approximation $\sin\alpha \approx \alpha$ is justified. For that case, write down the equation of motion for α as well as its general solution.