Hochschule RheinMain
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3. Homework Assignment Advanced Classical Mechanics

Exercise 1: We consider the motion of a particle in the (x, y)-plane and parametrize the motion by polar coordinates,

$$\vec{x}(t) \;=\; \begin{pmatrix} x_t \\ y_t \end{pmatrix} \;=\; r_t \begin{pmatrix} \cos\varphi_t \\ \sin\varphi_t \end{pmatrix}$$

Furthermore, we define the vectors

$$\vec{e}_r \;=\; \begin{pmatrix} \cos\varphi_t \\ \sin\varphi_t \end{pmatrix}, \quad \vec{e}_\varphi \;=\; \begin{pmatrix} -\sin\varphi_t \\ \cos\varphi_t \end{pmatrix}$$

Show:

a) The vectors $\vec{e}_r$ and $\vec{e}_\varphi$ form an orthonormal basis.

b) We have

$$\begin{aligned} \dot{\vec{e}}_r &= \dot{\varphi}_t \, \vec{e}_\varphi \\ \dot{\vec{e}}_\varphi &= -\dot{\varphi}_t \, \vec{e}_r \end{aligned}$$

c) We have

$$\dot{\vec{x}} \;=\; \dot{r} \, \vec{e}_r \;+\; r \, \dot{\varphi} \, \vec{e}_\varphi$$

d) The kinetic energy is given by

$$T \;=\; \tfrac{m}{2} \, \dot{\vec{x}}^2 \;=\; \tfrac{m}{2} \left\{ \dot{r}^2 \;+\; r^2 \, \dot{\varphi}^2 \right\}$$

e) We have

$$\ddot{\vec{x}} \;=\; \left(\ddot{r} \;-\; r \, \dot{\varphi}^2 \right) \vec{e}_r \;+\; \left(2 \, \dot{r} \, \dot{\varphi} \;+\; r \, \ddot{\varphi} \right) \vec{e}_\varphi \;\; .$$

Exercise 2: We consider the motion of a free particle (no forces acting) in the (x, y)-plane and parametrize the motion by polar coordinates. So we consider the same setting as in Exercise 1. If no external forces act on the particle, the equation of motion is simply given by

$$m \, \ddot{\vec{x}} \;=\; 0 \tag{1}$$

By Exercise (1e), equation (1) is equivalent to the system

$$\begin{aligned} \ddot{r} \;-\; r \, \dot{\varphi}^2 &= 0 \qquad (2) \\ 2 \, \dot{r} \, \dot{\varphi} \;+\; r \, \ddot{\varphi} &= 0 \qquad (3) \end{aligned}$$

Now have another look at the material in `week3.pdf` on the course homepage, in particular part (b) of Theorem 2.1.2 on the last page.

a) Choose $(u, v) = (r, \varphi)$ as the generalized coordinates and write down the Lagrangian L (you already computed it in Exercise 1..) .

b) Derive the Euler-Lagrange equations and show that they are equivalent to the system (2) and (3).