Hochschule RheinMain
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Solutions Homework Assignment 4 Advanced Classical Mechanics

Exercise 1: For two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ with

$$\vec{a} \;=\; \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \;=:\; a_1\,\vec{e}_1 \,+\, a_2\,\vec{e}_2 \,+\, a_3\,\vec{e}_3$$

the cross product $\vec{a} \times \vec{b} \in \mathbb{R}^3$ was defined by

$$\vec{a} \times \vec{b} \;:=\; \det \begin{pmatrix} \vec{e}_1 & a_1 & b_1 \\ \vec{e}_2 & a_2 & b_2 \\ \vec{e}_3 & a_3 & b_3 \end{pmatrix} \;=\; \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

In particular, we therefore have

$$\vec{a} \times \vec{a} \;=\; \vec{0}\,.$$

a) We have

$$\begin{aligned} \tfrac{d}{dt}\,\vec{L} \;&=\; m\tfrac{d}{dt}\,\{\vec{x} \times \dot{\vec{x}}\} \\ &=\; m\{\dot{\vec{x}} \times \dot{\vec{x}} \,+\, \vec{x} \times \ddot{\vec{x}}\} \\ &=\; m\{\vec{0} \,+\, \vec{x} \times \vec{0}\} \;=\; \vec{0}\,. \end{aligned}$$

b) We now have

$$\begin{aligned} m\,\ddot{\vec{x}} \;&=\; \vec{F} \,+\, \vec{Z} \\ &=\; -mg\,\vec{e}_3 \,+\, \lambda\,\vec{e}_r \\ &=\; -mg\,\vec{e}_3 \,+\, \tfrac{\lambda}{r}\,r\vec{e}_r \\ &=\; -mg\,\vec{e}_3 \,+\, \tfrac{\lambda}{r}\,\vec{x} \end{aligned}$$

and hence

$$\begin{aligned} \tfrac{d}{dt}\,\vec{L} \;&=\; m\tfrac{d}{dt}\,\{\vec{x} \times \dot{\vec{x}}\} \\ &=\; m\{\dot{\vec{x}} \times \dot{\vec{x}} \,+\, \vec{x} \times \ddot{\vec{x}}\} \\ &=\; \vec{x} \times \{m\,\ddot{\vec{x}}\} \\ &=\; \vec{x} \times \{-mg\,\vec{e}_3 \,+\, \tfrac{\lambda}{r}\,\vec{x}\} \\ &=\; -mg\,\vec{x} \times \vec{e}_3 \,+\, \tfrac{\lambda}{r}\,\vec{x} \times \vec{x} \\ &=\; -mg\,\vec{x} \times \vec{e}_3 \end{aligned}$$

with the cross product

$$\vec{x} \times \vec{e}_3 \;=\; \det \begin{pmatrix} \vec{e}_1 & x & 0 \\ \vec{e}_2 & y & 0 \\ \vec{e}_3 & z & 1 \end{pmatrix} \;=\; \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

So we have

$$\tfrac{d}{dt}\, L_3 \;=\; 0\,.$$

c) In `week4.pdf` we derived the following formulas in equation (5):

$$\begin{aligned} \dot{\vec{e}}_r &= \dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; \dot\theta\, \vec{e}_\theta \\ \dot{\vec{e}}_\varphi &= -\dot\varphi \,\sin\theta\, \vec{e}_r \;-\; \dot\varphi \,\cos\theta\, \vec{e}_\theta \\ \dot{\vec{e}}_\theta &= \dot\varphi \,\cos\theta\, \vec{e}_\varphi \;-\; \dot\theta\, \vec{e}_r \end{aligned}$$

Here we only need the first of these equations and obtain

$$\begin{aligned} \vec{x} &= r\, \vec{e}_r \\ \dot{\vec{x}} &= \dot r\, \vec{e}_r \;+\; r\, \dot{\vec{e}}_r \\ &= \tfrac{\dot r}{r}\, r\vec{e}_r \;+\; r\left\{\dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; \dot\theta\, \vec{e}_\theta\right\} \\ &= \tfrac{\dot r}{r}\, \vec{x} \;+\; r\dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; r\dot\theta\, \vec{e}_\theta \end{aligned}$$

We can substitute this into $\vec{L}$,

$$\begin{aligned} \vec{L} &= \vec{x} \times \vec{p} \;=\; m\, \vec{x} \times \dot{\vec{x}} \\ &= m\, \vec{x} \times \left\{\tfrac{\dot r}{r}\, \vec{x} \;+\; r\dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; r\dot\theta\, \vec{e}_\theta\right\} \\ &= m\, \vec{x} \times \left\{r\dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; r\dot\theta\, \vec{e}_\theta\right\} \\ &= mr\, \vec{e}_r \times \left\{r\dot\varphi \,\sin\theta\, \vec{e}_\varphi \;+\; r\dot\theta\, \vec{e}_\theta\right\} \\ &= mr^2\dot\varphi \,\sin\theta\, \vec{e}_r \times \vec{e}_\varphi \;+\; mr^2\dot\theta\, \vec{e}_r \times \vec{e}_\theta \end{aligned}$$

We compute the cross products,

$$\vec{e}_r \times \vec{e}_\varphi \;=\; \det \begin{pmatrix} \vec{e}_1 & \cos\varphi\,\sin\theta & -\sin\varphi \\ \vec{e}_2 & \sin\varphi\,\sin\theta & \cos\varphi \\ \vec{e}_3 & \cos\theta & 0 \end{pmatrix} \;=\; \begin{pmatrix} -\cos\varphi\,\cos\theta \\ -\sin\varphi\,\cos\theta \\ +\sin\theta \end{pmatrix} \;=\; -\,\vec{e}_\theta$$

and

$$\vec{e}_r \times \vec{e}_\theta \;=\; \det \begin{pmatrix} \vec{e}_1 & \cos\varphi\,\sin\theta & \cos\varphi\,\cos\theta \\ \vec{e}_2 & \sin\varphi\,\sin\theta & \sin\varphi\,\cos\theta \\ \vec{e}_3 & \cos\theta & -\sin\theta \end{pmatrix} \;=\; \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \;=\; +\,\vec{e}_\varphi$$

Thus,

$$\begin{aligned} \vec{L} &= mr^2\dot\varphi \,\sin\theta\, \vec{e}_r \times \vec{e}_\varphi \;+\; mr^2\dot\theta\, \vec{e}_r \times \vec{e}_\theta \\ &= -\,mr^2\dot\varphi \,\sin\theta\, \vec{e}_\theta \;+\; mr^2\dot\theta\, \vec{e}_\varphi \end{aligned}$$

and, since the z-component of $\vec{e}_\varphi$ is zero and that of $\vec{e}_\theta$ is $-\sin\theta$,

$$L_3 \;=\; +\,m\,r^2\,\dot\varphi\,\sin^2\theta\,.$$

Exercise 2: We have the equations of motion

$$\begin{aligned} \tfrac{d}{dt}\big\{\,\dot\varphi \sin^2\theta\,\big\} &= 0 \\ \ddot\theta \;-\; \dot\varphi^2 \sin\theta\cos\theta \;-\; \tfrac{g}{R}\sin\theta &= 0 \end{aligned}$$

or, if c denotes a time-independent constant,

$$\dot\varphi \sin^2\theta = c \tag{1}$$

$$\ddot\theta \;-\; \dot\varphi^2 \sin\theta\cos\theta \;-\; \tfrac{g}{R}\sin\theta = 0 \tag{2}$$

a) If the ball is to move on horizontal circles, then the angle θ must be constant,

$$\theta_t = \theta_0 \tag{3}$$

In particular, the first and second derivatives of θ are then zero,

$$\dot\theta = \ddot\theta = 0$$

Equations (1) and (2) thus become

$$\dot\varphi \sin^2\theta_0 = c \tag{4}$$

$$-\,\dot\varphi^2 \sin\theta_0\cos\theta_0 \;-\; \tfrac{g}{R}\sin\theta_0 = 0 \tag{5}$$

On the one hand, we have the special solution $\sin\theta_0 = 0$, i.e. $\theta_0 = \pi$, where the ball simply rests at the lowest point of the bowl at $\vec{x} = (0, 0, -R)$ (the unstable solution $\theta = 0$, i.e. $\vec{x} = (0, 0, +R)$, one might perhaps also list for a rigid spherical pendulum, but we said that here we have a bowl..).

For $\sin\theta_0 \neq 0$ we can divide by $\sin\theta_0$ or $\sin^2\theta_0$ and obtain

$$\dot\varphi = \tfrac{c}{\sin^2\theta_0} =: \omega_0 \tag{6}$$

$$-\,\dot\varphi^2 \cos\theta_0 \;-\; \tfrac{g}{R} = 0 \tag{7}$$

So the particle must move with a constant angular velocity $\dot\varphi_0 = \omega_0$, and if the particle is to move on a circle of latitude with angle θ_0, then the angular velocity ω_0 must be chosen as follows:

$$\omega_0^2 \cos\theta_0 \;+ g/R = 0$$

or

$$\omega_0 = \sqrt{\frac{g}{R(-\cos\theta_0)}} \tag{8}$$

Note that, because $\theta_0 \in [\pi/2, \pi]$, the quantity $-\cos\theta_0$ in the denominator of (8) is positive.

b) For $\dot{\varphi} = 0$ we obtain from equation (2)

$$\ddot{\theta} \; - \; \tfrac{g}{R} \sin\theta \;\; = \;\; 0 \tag{9}$$

With the substitution $\theta = \pi - \alpha$ we obtain

$$\ddot{\theta} \;\; = \;\; -\ddot{\alpha}$$

and

$$\sin\theta \;\; = \;\; \sin(\pi - \alpha) \;\; = \;\; \sin\alpha$$

Substituting this into (9), we thus obtain

$$-\ddot{\alpha} \; - \; \tfrac{g}{R} \sin\alpha \;\; = \;\; 0$$

or

$$\ddot{\alpha} \; + \; \tfrac{g}{R} \sin\alpha \;\; = \;\; 0 \tag{10}$$

Now we consider the case of small displacements $\alpha \ll 1$, so that we can make the approximation

$$\sin\alpha \;\; = \;\; \alpha \; - \; \tfrac{\alpha^3}{3!} \; + \; \tfrac{\alpha^5}{5!} \; \mp \cdots \;\; \approx \;\; \alpha$$

From (10) we then obtain

$$\ddot{\alpha} \; + \; \tfrac{g}{R} \, \alpha \;\; = \;\; 0 \tag{11}$$

Equation (11) is the differential equation of the harmonic oscillator. As a linear second-order differential equation, it has 2 linearly independent solutions; these are given by $e^{\pm i\omega t}$ or, if we make it real right away, by $\sin\omega t$ and $\cos\omega t$. Substituting this into (11), one obtains for ω the equation

$$-\omega^2 \; + \; \tfrac{g}{R} \;\; = \;\; 0$$

or $\omega = \pm\sqrt{\frac{g}{R}}$. Because $\cos(-\omega t) = \cos(\omega t)$ and $\sin(-\omega t) = -\sin(\omega t)$, the negative ω does not give any additional new solutions, so we take

$$\omega \;\; = \;\; \sqrt{\tfrac{g}{R}} \;\; =: \;\; \omega_0 \tag{12}$$

and then obtain the general solution

$$\alpha(t) \;\; = \;\; c_1 \, \cos(\omega_0 t) \; + \; c_2 \, \sin(\omega_0 t) \tag{13}$$

As usual, the constants c_1 and c_2 are determined from the initial conditions, i.e. the initial displacement α_0 and the initial angular velocity $\dot{\alpha}_0$:

$$\alpha_0 \; = \; \alpha(0) \;\; = \;\; c_1 \, \cos(0) \; + \; c_2 \, \sin(0)$$

$$\dot{\alpha}_0 \; = \; \dot{\alpha}(0) \;\; = \;\; -c_1 \, \omega_0 \, \sin(0) \; + \; c_2 \, \omega_0 \, \cos(0)$$

or

$$\begin{aligned} \alpha_0 \;\; &= \;\; c_1 \\ \dot{\alpha}_0 \;\; &= \;\; c_2 \, \omega_0 \end{aligned}$$

This gives us the final result

$$\alpha(t) \;\; = \;\; \alpha_0 \, \cos(\omega_0 t) \; + \; \tfrac{\dot{\alpha}_0}{\omega_0} \, \sin(\omega_0 t) \;\; . \tag{14}$$