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Solutions Homework Assignment 3 Advanced Classical Mechanics

Exercise 1: We had

$$\vec{x}(t) \;=\; \begin{pmatrix} x_t \\ y_t \end{pmatrix} \;=\; r_t \begin{pmatrix} \cos\varphi_t \\ \sin\varphi_t \end{pmatrix}$$

and

$$\vec{e}_r \;=\; \begin{pmatrix} \cos\varphi_t \\ \sin\varphi_t \end{pmatrix}, \quad \vec{e}_\varphi \;=\; \begin{pmatrix} -\sin\varphi_t \\ \cos\varphi_t \end{pmatrix}$$

a) We have

$$\|\vec{e}_r\|^2 \;=\; \|\vec{e}_\varphi\|^2 \;=\; \cos^2\varphi + \sin^2\varphi \;=\; 1$$

and

$$\vec{e}_r \cdot \vec{e}_\varphi \;=\; -\cos\varphi\,\sin\varphi \,+\, \cos\varphi\,\sin\varphi \;=\; 0$$

b) We have

$$\begin{aligned} \dot{\vec{e}}_r \;&=\; \dot{\varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} \;=\; \dot{\varphi}\,\vec{e}_\varphi \\ \dot{\vec{e}}_\varphi \;&=\; \dot{\varphi} \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \end{pmatrix} \;=\; -\dot{\varphi}\,\vec{e}_r \end{aligned}$$

c) We have

$$\vec{x} \;=\; r\,\vec{e}_r$$

and hence

$$\begin{aligned} \dot{\vec{x}} \;&=\; \dot{r}\,\vec{e}_r \,+\, r\,\dot{\vec{e}}_r \\ &=\; \dot{r}\,\vec{e}_r \,+\, r\,\dot{\varphi}\,\vec{e}_\varphi \end{aligned}$$

d) Since the vectors $\{\vec{e}_r, \vec{e}_\varphi\}$ form an orthonormal basis, we obtain

$$\begin{aligned} \dot{\vec{x}}^2 \;&=\; \left(\dot{r}\,\vec{e}_r \,+\, r\,\dot{\varphi}\,\vec{e}_\varphi\right)^2 \\ &=\; \dot{r}^2\,\vec{e}_r^{\,2} \,+\, 2\,\dot{r}\,r\,\dot{\varphi}\,\vec{e}_r\vec{e}_\varphi \,+\, r^2\,\dot{\varphi}^2\,\vec{e}_\varphi^{\,2} \\ &=\; \dot{r}^2 \,+\, r^2\,\dot{\varphi}^2 \end{aligned}$$

e) From part (c) it follows that

$$\begin{aligned}
\ddot{\vec{x}} &= \tfrac{d}{dt}\dot{\vec{x}} = \tfrac{d}{dt}\big\{\dot r\,\vec e_r + r\,\dot\varphi\,\vec e_\varphi\big\} \\
&= \ddot r\,\vec e_r + \dot r\,\dot{\vec e}_r + \dot r\,\dot\varphi\,\vec e_\varphi + r\,\ddot\varphi\,\vec e_\varphi + r\,\dot\varphi\,\dot{\vec e}_\varphi \\
&\overset{(b)}{=} \ddot r\,\vec e_r + \dot r\,\dot\varphi\,\vec e_\varphi + \dot r\,\dot\varphi\,\vec e_\varphi + r\,\ddot\varphi\,\vec e_\varphi + r\,\dot\varphi\,(-\dot\varphi\,\vec e_r) \\
&= \big(\ddot r - r\,\dot\varphi^2\big)\,\vec e_r + \big(2\,\dot r\,\dot\varphi + r\,\ddot\varphi\big)\,\vec e_\varphi\,.
\end{aligned}$$

Exercise 2: **a)** The Lagrangian L is given by

$$\begin{aligned}
L &= \text{kinetic energy} - \text{potential energy} \\
&= T - V \\
&\overset{V=0}{=} T \\
&= \tfrac{m}{2}\,\dot{\vec x}^2 \\
&\overset{(1d)}{=} \tfrac{m}{2}\,\big\{\,\dot r^2 + r^2\,\dot\varphi^2\,\big\}
\end{aligned}$$

b) The Euler-Lagrange equations read

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot r} - \tfrac{\partial L}{\partial r} = 0 \tag{1}$$

$$\tfrac{d}{dt}\tfrac{\partial L}{\partial \dot\varphi} - \tfrac{\partial L}{\partial \varphi} = 0 \tag{2}$$

Obviously,

$$\begin{aligned}
\tfrac{\partial L}{\partial r} &= m r\dot\varphi^2 \\
\tfrac{\partial L}{\partial \dot r} &= m\dot r \\
\tfrac{\partial L}{\partial \varphi} &= 0 \\
\tfrac{\partial L}{\partial \dot\varphi} &= m r^2\dot\varphi
\end{aligned}$$

Hence equation (1) is equivalent to

$$m\ddot r - m r\dot\varphi^2 = 0 \tag{3}$$

and equation (2) is equivalent to

$$\tfrac{d}{dt}\big\{m r^2\dot\varphi\big\} = m\big\{2r\dot r\dot\varphi + r^2\ddot\varphi\big\} = 0 \tag{4}$$

Dividing equation (3) by m and equation (4) by mr, we thus obtain the system

$$\begin{aligned}
\ddot r - r\,\dot\varphi^2 &= 0 \\
2\,\dot r\,\dot\varphi + r\,\ddot\varphi &= 0\,.
\end{aligned}$$