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Solutions Homework Assignment 2 Advanced Classical Mechanics

Exercise 1: We have

$$\begin{aligned}
\tfrac{d}{dt} e^{-\frac{1}{2}\int_0^t \gamma_s\, ds} &= -\tfrac{\gamma_t}{2}\, e^{-\frac{1}{2}\int_0^t \gamma_s\, ds} \\
\tfrac{d^2}{dt^2} e^{-\frac{1}{2}\int_0^t \gamma_s\, ds} &= -\tfrac{\dot\gamma_t}{2}\, e^{-\frac{1}{2}\int_0^t \gamma_s\, ds} + \tfrac{\gamma_t^2}{4}\, e^{-\frac{1}{2}\int_0^t \gamma_s\, ds}
\end{aligned}$$

and the following formula holds

$$\tfrac{d^2}{dt^2}(f\cdot g) = \tfrac{d^2 f}{dt^2}\cdot g + 2\tfrac{df}{dt}\cdot\tfrac{dg}{dt} + f\cdot\tfrac{d^2 g}{dt^2}$$

With the abbreviation

$$\Gamma_t := e^{-\frac{1}{2}\int_0^t \gamma_s\, ds}$$

we then obtain

$$\begin{aligned}
\dot x_t &= \tfrac{d}{dt}(\Gamma_t\cdot y_t) = -\tfrac{\gamma_t}{2}\,\Gamma_t\cdot y_t + \Gamma_t\cdot\dot y_t \\
\ddot x_t &= \tfrac{d^2}{dt^2}(\Gamma_t\cdot y_t) \\
&= \tfrac{d^2}{dt^2}\Gamma_t\cdot y_t + 2\,\tfrac{d}{dt}\Gamma_t\cdot\dot y_t + \Gamma_t\cdot\ddot y_t \\
&= -\tfrac{\dot\gamma_t}{2}\,\Gamma_t\cdot y_t + \tfrac{\gamma_t^2}{4}\,\Gamma_t\cdot y_t - \gamma_t\,\Gamma_t\cdot\dot y_t + \Gamma_t\cdot\ddot y_t
\end{aligned}$$

and hence

$$\begin{aligned}
\ddot x_t + \gamma_t\dot x_t + \omega^2 x_t &= -\tfrac{\dot\gamma_t}{2}\,\Gamma_t\cdot y_t + \tfrac{\gamma_t^2}{4}\,\Gamma_t\cdot y_t - \gamma_t\,\Gamma_t\cdot\dot y_t + \Gamma_t\cdot\ddot y_t \\
&\quad + \gamma_t\Big\{-\tfrac{\gamma_t}{2}\,\Gamma_t\cdot y_t + \Gamma_t\cdot\dot y_t\Big\} + \omega^2\,\Gamma_t\cdot y_t \\
&= \Gamma_t\cdot y_t\Big\{-\tfrac{\dot\gamma_t}{2} + \tfrac{\gamma_t^2}{4} - \tfrac{\gamma_t^2}{2} + \omega^2\Big\} \\
&\quad - \gamma_t\,\Gamma_t\cdot\dot y_t + \gamma_t\,\Gamma_t\cdot\dot y_t + \Gamma_t\cdot\ddot y_t \\
&= \Gamma_t\Big(\ddot y_t + \Big\{\omega^2 - \tfrac{\gamma_t^2}{4} - \tfrac{\dot\gamma_t}{2}\Big\}\, y_t\Big)
\end{aligned}$$

Since $\Gamma_t \neq 0$, this implies the equivalence to

$$\ddot y_t + \omega_t^2\, y_t = 0$$

with

$$\omega_t^2 \;=\; \omega^2 \,-\, \tfrac{\gamma_t^2}{4} \,-\, \tfrac{\dot\gamma_t}{2} \;\;.$$

Exercise 2: a) We have

$$\begin{aligned} \gamma_t &= 2\mu \sin(\mu t) \\ \textstyle\int_0^t \gamma_s \, ds &= -2\,\cos(\mu s)\big|_0^t \;=\; 2[1-\cos(\mu t)] \end{aligned}$$

so the damping factor is given by

$$\Gamma_t \;=\; e^{-\frac{1}{2}\int_0^t \gamma_s\, ds} \;=\; e^{-[1-\cos(\mu t)]}$$

b) We obtain

$$\begin{aligned} \omega_t^2 &= \omega^2 \,-\, \tfrac{\gamma_t^2}{4} \,-\, \tfrac{\dot\gamma_t}{2} \\ &= 1 \,-\, \mu^2 \,\sin^2(\mu t) \,-\, \mu^2 \,\cos(\mu t) \end{aligned}$$

c) Since

$$\mu^2 \;=\; \frac{1}{50^2} \;=\; \frac{1}{2500} \;=\; \frac{4}{10000} \;=\; 0.0004$$

$\omega_t^2 \approx 1$ is obviously a plausible approximation. The ODE for y_t then reads

$$\ddot y_t \,+\, y_t \;=\; 0$$

and, for the initial conditions $y_0 = 1$ and $\dot y_0 = 0$, has the solution

$$y_t \;=\; \cos t$$

Hence

$$x_t \;=\; e^{-[1-\cos(\mu t)]} \cdot \cos t$$

is an approximate solution of the given ODE which also satisfies the initial conditions $x_0 = 1$ and $\dot x_0 = 0$.