

week2: Matrix-Exponential and Harmonic Oscillator

Let us first recall the power series expansions of exp, cos and sin which are immediate consequences of the general Taylor formula:

Theorem 2.1 (Taylor expansion): Let $f = f(x) : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function and let $x_0 \in \mathbb{R}$ be a fixed expansion point. Then, under mild¹ technical assumptions:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \end{aligned} \quad (2.1)$$

Here $f^{(k)}(x_0)$ denotes the k -th derivative of f at the point x_0 , and the expression $k!$, k factorial, is given by

$$k! := 1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k \quad (2.2)$$

We use the conventions $f^{(0)}(x) := f(x)$ and $0! := 1$.

Proof: This is one of the most important formulae in mathematics, in particular for physicists and engineers, so we attach a proof for it which can be found in appendix 1. ■

From this theorem we immediately obtain the power series expansions for sine, cosine and the exponential function by choosing $x_0 = 0$:

Theorem 2.2: For arbitrary $x \in \mathbb{R}$ the following power series expansions hold:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (2.3)$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (2.4)$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (2.5)$$

¹the expression $R_n := \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$ has to converge to 0 for $n \rightarrow \infty$; for sin, cos and exp this is the case

Definition 2.1: Let A be an arbitrary real or complex $n \times n$ matrix. Then the matrix-exponential e^A is defined through

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} = Id + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots \quad (2.6)$$

where Id denotes the $n \times n$ unit matrix with 1's on the diagonal and zeros otherwise.

One of the major applications of the matrix exponential is that systems of ordinary differential equations with constant coefficients can be solved with it, due to the following theorem:

Theorem 2.3: Let $A \in \mathbb{R}^{n \times n}$ be an arbitrary constant (i.e. not depending on the time t) matrix and $\vec{y} = \vec{y}(t) \in \mathbb{R}^n$. Then: The differential equation

$$\vec{y}'(t) = A \vec{y}(t)$$

is solved by

$$\vec{y}(t) = e^{tA} \vec{y}_0 \quad (2.7)$$

with $\vec{y}_0 := \vec{y}(t=0)$.

Proof: We have

$$\begin{aligned} \frac{d}{dt} e^{At} &= \frac{d}{dt} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} A^n \frac{d}{dt} \frac{t^n}{n!} = \sum_{n=1}^{\infty} A^n \frac{n t^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} A^n \frac{t^{n-1}}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{(At)^{n-1}}{(n-1)!} = A \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = A e^{At} \end{aligned}$$

With $\vec{y}(t) := e^{At} \vec{y}_0$ we then obtain

$$\vec{y}'(t) = \frac{d\vec{y}}{dt} = \frac{d}{dt} e^{At} \vec{y}_0 = A e^{At} \vec{y}_0 = A \vec{y}(t)$$

and the theorem is proven. ■

Example: Damped harmonic Oscillator

Let us recall the differential equation for the damped (or, for $\gamma = 0$, undamped) harmonic oscillator. It was given by

$$x''(t) + \gamma x'(t) + \varepsilon^2 x(t) = 0 \quad (2.8)$$

We want to consider the velocity $v = v(t) = x'(t)$ as an independent variable and write the second derivative $x''(t)$ as the first derivative of v ,

$$v(t) := x'(t) \Rightarrow v'(t) = x''(t) \quad (2.9)$$

We can then write

$$x''(t) + \gamma x'(t) + \varepsilon^2 x(t) = v'(t) + \gamma v(t) + \varepsilon^2 x(t) = 0 \quad (2.10)$$

and the original second-order differential equation (2.8) is then equivalent to the first-order system

$$\begin{aligned}x'(t) &= v(t) \\v'(t) &= -\gamma v(t) - \varepsilon^2 x(t)\end{aligned}\tag{2.11}$$

We can now write this in vector-matrix notation as follows:

$$\begin{pmatrix} x'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & -\gamma \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}\tag{2.12}$$

If we then define the 2-component vector

$$\vec{y}(t) := \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}\tag{2.13}$$

with derivative

$$\begin{aligned}\vec{y}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\vec{y}(t + \Delta t) - \vec{y}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\begin{pmatrix} x(t + \Delta t) \\ v(t + \Delta t) \end{pmatrix} - \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \right] \\&= \lim_{\Delta t \rightarrow 0} \begin{pmatrix} \frac{x(t + \Delta t) - x(t)}{\Delta t} \\ \frac{v(t + \Delta t) - v(t)}{\Delta t} \end{pmatrix} = \begin{pmatrix} x'(t) \\ v'(t) \end{pmatrix} \stackrel{\text{Notation}}{=} \frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix}(t)\end{aligned}\tag{2.14}$$

then the system (2.12) is equivalent to the ODE

$$\vec{y}'(t) = A \vec{y}(t)\tag{2.15}$$

with 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & -\gamma \end{pmatrix}\tag{2.16}$$

According to Theorem 3.3 we can then write down a solution in terms of the matrix exponential:

$$\begin{pmatrix} x \\ v \end{pmatrix}(t) = \exp \left[\begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & -\gamma \end{pmatrix} t \right] \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}\tag{2.17}$$

How can the matrix exponential e^{At} actually be computed? One method is to diagonalize the matrix A ; we will look at this shortly in Theorem 2.5. This is a systematic method which always works, provided the given matrix A is diagonalizable. Sometimes, however, as in the present case, the given matrix A is simple enough such that the power series expansion can be summed up explicitly. In order to do this here, we still need the following

Lemma 2.1: Let $A, B \in \mathbb{R}^{n \times n}$ be two arbitrary $n \times n$ matrices with

$$AB = BA\tag{2.18}$$

Then:

$$e^{A+B} = e^A e^B\tag{2.19}$$

Remark: For matrices, the standard case is typically that assumption (2.18) is not satisfied, so that equation (2.19) does not hold either.

Proof: is completely analog to the proof of $e^{z+w} = e^z e^w$ for complex numbers z and w , this proof is given in the appendix. There then the complex numbers z and w now have to be replaced by the matrices A and B . The assumption $AB = BA$ is needed in order to be able to apply the binomial formula (2.41) for $(a+b)^n$ to matrices $(A+B)^n$ as well. ■

With this we can now compute the matrix exponential in (2.17). We put $\gamma := 2\mu$.

Theorem 2.4: With $\omega := \sqrt{\varepsilon^2 - \mu^2}$ the following formula holds:

$$\exp\left\{t \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & -2\mu \end{pmatrix}\right\} = e^{-\mu t} \left[\cos \omega t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin \omega t}{\omega} \begin{pmatrix} +\mu & +1 \\ -\varepsilon^2 & -\mu \end{pmatrix} \right] \quad (2.20)$$

Proof: We write

$$\begin{aligned} \exp\left\{t \begin{pmatrix} 0 & 1 \\ -\varepsilon^2 & -2\mu \end{pmatrix}\right\} &= \exp\left\{t \begin{pmatrix} -\mu & 0 \\ 0 & -\mu \end{pmatrix} + t \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix}\right\} \\ &= \exp\left\{-\mu t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix}\right\} \\ &\stackrel{(2.19)}{=} \exp\left\{-\mu t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \exp\left\{t \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix}\right\} \\ &= e^{-\mu t} Id \cdot \exp\left\{t \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix}\right\} \end{aligned}$$

Furthermore, with

$$A := \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix}$$

we obtain the following powers:

$$A^2 = \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix} \begin{pmatrix} +\mu & 1 \\ -\varepsilon^2 & -\mu \end{pmatrix} = \begin{pmatrix} \mu^2 - \varepsilon^2 & 0 \\ 0 & \mu^2 - \varepsilon^2 \end{pmatrix} = -\omega^2 Id$$

and hence

$$A^{2k} = (A^2)^k = (-1)^k \omega^{2k} Id \quad (2.21)$$

$$A^{2k+1} = A^{2k} \cdot A \stackrel{(2.21)}{=} (-1)^k \omega^{2k} A = (-1)^k \omega^{2k+1} \frac{A}{\omega} \quad (2.22)$$

With the power series expansions for exp, cos and sin we then obtain:

$$\begin{aligned} e^{tA} &\stackrel{(2.6)}{=} \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \sum_{k=0}^{\infty} \frac{(tA)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(tA)^{2k+1}}{(2k+1)!} \\ &\stackrel{(2.21)}{=} \sum_{k=0}^{\infty} (-1)^k \frac{(t\omega)^{2k}}{(2k)!} Id + \sum_{k=0}^{\infty} (-1)^k \frac{(t\omega)^{2k+1}}{(2k+1)!} \frac{A}{\omega} \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.4)}{=} \stackrel{(2.5)}{=} \cos(\omega t) Id + \frac{\sin(\omega t)}{\omega} A \\
& = \cos \omega t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin \omega t}{\omega} \begin{pmatrix} +\mu & +1 \\ -\varepsilon^2 & -\mu \end{pmatrix}
\end{aligned}$$

This proves the formula. ■

As the solution for the damped harmonic oscillator we therefore obtain

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = e^{-\mu t} \begin{pmatrix} \cos \omega t + \frac{\mu}{\omega} \sin \omega t & \frac{\sin \omega t}{\omega} \\ -\varepsilon^2 \frac{\sin \omega t}{\omega} & \cos \omega t - \frac{\mu}{\omega} \sin \omega t \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \quad (2.23)$$

We are interested in the displacement $x(t)$, which is the first coordinate of this, and we obtain

$$\begin{aligned}
x(t) &= e^{-\mu t} \left[\left(\cos \omega t + \frac{\mu}{\omega} \sin \omega t \right) x_0 + \frac{\sin \omega t}{\omega} v_0 \right] \\
&= e^{-\mu t} \left[x_0 \cos \omega t + \frac{v_0 + \mu x_0}{\omega} \sin \omega t \right] \quad (2.24)
\end{aligned}$$

For those people who also attend the course on Advanced Classical Mechanics: This is the same formula we also obtained already in exercise (1b) of the first assignment sheet.

For diagonalizable matrices the matrix exponential can be computed systematically as follows:

Theorem 2.5: Let A be a real or complex diagonalizable $n \times n$ matrix with a basis of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Let V be the matrix of eigenvectors, more precisely

$$V := \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \quad (2.25)$$

and let V^{-1} be the inverse matrix of V , so that

$$A = V D V^{-1} \quad (2.26)$$

with the diagonal matrix of eigenvalues

$$D := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} =: \text{Diag}(\lambda_1, \dots, \lambda_n) \quad (2.27)$$

Then

$$e^{tA} = V e^{tD} V^{-1} \quad (2.28)$$

with

$$e^{tD} = \text{Diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n}) \quad (2.29)$$

Proof: We write down a proof for $n = 2$; the general case is completely analogous. We have the matrix products

$$\begin{aligned}
D^2 &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \\
D^3 &= D^2 \cdot D = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} \\
&\vdots \\
D^n &= D^{n-1} \cdot D = \begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{tD} &= \sum_{n=0}^{\infty} \frac{(tD)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{t^n}{n!} \lambda_1^n & 0 \\ 0 & \frac{t^n}{n!} \lambda_2^n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(t\lambda_1)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(t\lambda_2)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}
\end{aligned}$$

With

$$A = V D V^{-1}$$

we then obtain

$$\begin{aligned}
A^2 &= V D \underbrace{V^{-1} V}_{= Id} D V^{-1} = V D Id D V^{-1} = V D^2 V^{-1} \\
A^3 &= A^2 \cdot A = V D^2 \underbrace{V^{-1} V}_{= Id} D V^{-1} = V D^2 Id D V^{-1} = V D^3 V^{-1} \\
&\vdots \\
A^n &= A^{n-1} \cdot A = V D^{n-1} V^{-1} V D V^{-1} = V D^{n-1} Id D V^{-1} = V D^n V^{-1}
\end{aligned}$$

and arrive at

$$\begin{aligned}
e^{tA} &= \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} V D^n V^{-1} = \sum_{n=0}^{\infty} V \frac{t^n}{n!} D^n V^{-1} \\
&= V \left[\sum_{n=0}^{\infty} \frac{(tD)^n}{n!} \right] V^{-1} = V e^{tD} V^{-1} = V \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} V^{-1}
\end{aligned}$$

which proves the theorem. ■

A typical application of this theorem is given in the first exercise of the new exercise sheet 2.

Appendix 1: Proof of the Taylor-Formula

Due to its importance, let us recall here the proof of the Taylor series expansion formula:

Proof of Theorem 2.1: Although the theorem is very important and is used very frequently, the derivation of this formula is not particularly difficult; all we need is the formula for integration by parts, so let us quickly recall it: By the product rule for differentiating a product $g \cdot h$ of two functions $g = g(t)$ and $h = h(t)$ we have

$$(g \cdot h)'(t) = g'(t) h(t) + g(t) h'(t)$$

We integrate this from x_0 to x and obtain

$$(g \cdot h)(x) - (g \cdot h)(x_0) = \int_{x_0}^x (g \cdot h)'(t) dt = \int_{x_0}^x g'(t) h(t) dt + \int_{x_0}^x g(t) h'(t) dt$$

or

$$\begin{aligned} \int_{x_0}^x g(t) h'(t) dt &= (g \cdot h)(x) - (g \cdot h)(x_0) - \int_{x_0}^x g'(t) h(t) dt \\ &= (g \cdot h)(t) \Big|_{t=x_0}^{t=x} - \int_{x_0}^x g'(t) h(t) dt \end{aligned} \quad (2.30)$$

To derive the Taylor formula, we start as follows:

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt = f(x_0) + \int_{x_0}^x f'(t) \cdot 1 dt \quad (2.31)$$

To the integral on the right-hand side of (2.31) we now apply formula (2.30) with

$$\begin{aligned} g(t) &:= f'(t) \\ h'(t) &:= 1 \end{aligned}$$

If $h'(t) = 1$ is to hold, we can take $h(t) = t$. However, we could just as well choose $h(t) = t - 1$ or $h(t) = t + 2$ or, in general, $h(t) = t + c$ with a constant c , since constants vanish under differentiation. In order to derive the Taylor formula, it is now convenient to choose

$$h(t) = t - x$$

In $h'(t)$ we differentiate with respect to t , not with respect to x , so x can be regarded as a constant, just like a 2 or a c . Substituting this into (2.30), we obtain

$$\begin{aligned} \int_{x_0}^x f'(t) \cdot 1 dt &= [f'(t) \cdot (t - x)]_{t=x_0}^{t=x} - \int_{x_0}^x f''(t) (t - x) dt \\ &= 0 - f'(x_0) \cdot (x_0 - x) - \int_{x_0}^x f''(t) (t - x) dt \\ &= f'(x_0) \cdot (x - x_0) + \int_{x_0}^x f''(t) (x - t) dt \end{aligned} \quad (2.32)$$

We substitute this into (2.31) and obtain

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^x f'(t) \cdot 1 dt \\ &= f(x_0) + f'(x_0) \cdot (x - x_0) + \int_{x_0}^x f''(t) (x - t) dt \end{aligned} \quad (2.33)$$

Let us do one more iteration. To the integral on the right-hand side of (2.33) we again apply formula (2.30), now with

$$\begin{aligned} g(t) &:= f''(t) \\ h'(t) &:= x - t \end{aligned}$$

As an antiderivative of h' we can then take

$$h(t) = -\frac{1}{2}(x-t)^2 = -\frac{1}{2!}(x-t)^2$$

With (2.30) we then obtain

$$\begin{aligned} \int_{x_0}^x f''(t)(x-t) dt &= (g \cdot h)|_{t=x_0}^{t=x} - \int_{x_0}^x g'(t)h(t) dt \\ &= -\frac{1}{2!} [f''(t) \cdot (x-t)^2]_{t=x_0}^{t=x} + \frac{1}{2!} \int_{x_0}^x f'''(t)(x-t)^2 dt \\ &= 0 + \frac{1}{2!} f''(x_0) \cdot (x-x_0)^2 + \frac{1}{2!} \int_{x_0}^x f'''(t)(x-t)^2 dt \end{aligned} \quad (2.34)$$

and substituting this into (2.33),

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0) \cdot (x-x_0) + \int_{x_0}^x f''(t)(x-t) dt \\ &\stackrel{(2.34)}{=} f(x_0) + f'(x_0) \cdot (x-x_0) + \frac{1}{2!} f''(x_0) \cdot (x-x_0)^2 + \frac{1}{2!} \int_{x_0}^x f'''(t)(x-t)^2 dt \end{aligned} \quad (2.35)$$

Now one can see how the Taylor formula gradually builds up, and after n iterations one obtains

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0) \cdot (x-x_0) + \frac{1}{2!} f''(x_0) \cdot (x-x_0)^2 + \dots \\ &\quad \dots + \frac{1}{n!} f^{(n)}(x_0) \cdot (x-x_0)^n + \underbrace{\frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt}_{=: R_n(x)} \end{aligned} \quad (2.36)$$

Under the assumption that the remainder term $R_n(x)$ converges to zero as n becomes very large, which it does for the functions $\sin x$, $\cos x$ and $\exp x = e^x$, we thus obtain the Taylor series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) (x-x_0)^k \quad (2.37)$$

This proves the theorem. ■

Appendix 2: Proof of Lemma 2.1

Theorem: For arbitrary complex numbers $z, w \in \mathbb{C}$ we have

$$e^{z+w} = e^z \cdot e^w \quad (2.38)$$

Proof: We use the power series expansion of the exponential function:

$$\begin{aligned} e^z \cdot e^w &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \sum_{\ell=0}^{\infty} \frac{w^\ell}{\ell!} = \sum_{k,\ell=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^\ell}{\ell!} = \sum_{n=0}^{\infty} \sum_{\substack{k,\ell=0 \\ k+\ell=n}}^{\infty} \frac{z^k}{k!} \cdot \frac{w^\ell}{\ell!} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{k,\ell=0 \\ k+\ell=n}}^n \frac{z^k}{k!} \cdot \frac{w^\ell}{\ell!} \stackrel{\ell=n-k}{=} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} \end{aligned} \quad (2.39)$$

with the binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k} \quad (2.40)$$

Now the following formulas hold for real or complex a, b ; we only use addition and multiplication, and for these the same arithmetic rules apply to real and to complex numbers:

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

or in general

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{2}a^2b^{n-2} + nab^{n-1} + b^n \\&= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\&= \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}\end{aligned}\tag{2.41}$$

But this is exactly the expression we have above in (2.39). Hence we obtain

$$e^z \cdot e^w \stackrel{(2.39)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} \stackrel{(2.41)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n \stackrel{(2.3)}{=} e^{z+w}$$

and the statement is proven. ■