

**2. Exercise Sheet Advanced Mathematics
for Applied Physics and Biomedical Engineering**

Exercise 1) Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

- a) Determine the eigenvalues of A .
- b) Determine an orthonormal basis of eigenvectors and check the equation

$$A = V D V^T \tag{1}$$

through explicit calculation. Here V is the matrix of eigenvectors (the columns) and D is the diagonal matrix with the eigenvalues on the diagonal.

- c) Compute the matrix exponential e^{tA} by using equation (1).
- d) Solve the ODE system

$$\begin{aligned} \dot{x} &= x + 3y \\ \dot{y} &= 3x + y \end{aligned}$$

with initial conditions $x_{t=0} = 1$ and $y_{t=0} = 0$.

Exercise 2) Let

$$A = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

Show the following identity:

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ +\sin t & \cos t \end{pmatrix}$$

To do so, compute the even and odd powers A^{2k} and A^{2k+1} and use the power series of $\exp$, $\cos$ and $\sin$.

Exercise 3) The hyperbolic functions $\sinh x$ and $\cosh x$ are defined by

$$\cosh x := \frac{e^x + e^{-x}}{2}, \quad \sinh x := \frac{e^x - e^{-x}}{2} \tag{2}$$

- a) Show, using the power series expansion of the exponential function (or using the Taylor formula), that

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad (3)$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (4)$$

- b) Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Show the following identity:

$$e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

To do so, again compute the even and odd powers A^{2k} and A^{2k+1} and use the power series of $\exp$, $\cosh$ and $\sinh$.

Exercise 4) We want to check the power series expansions for $\sin$, $\cos$ and $\exp$ numerically using the R software. To this end, let

$$e_n(x) := \sum_{j=0}^n \frac{x^j}{j!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$s_n(x) := \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$c_n(x) := \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!}$$

so that

$$\lim_{n \rightarrow \infty} \begin{cases} e_n(x) \\ s_n(x) \\ c_n(x) \end{cases} = \begin{cases} e^x \\ \sin x \\ \cos x \end{cases}$$

- a) Using the R software, plot the functions $e_n(x)$ and e^x in an (x, y) -diagram for the values $n = 1, 2, \dots, 10$. Choose, say, $x \in [-2, +2]$.
- b) Plot the functions $s_n(x)$ and $\sin x$ in an (x, y) -diagram for the values $n = 1, 2, \dots, 10$. Choose for example $x \in [-2\pi, +2\pi]$ and $y \in [-2, +2]$.
- c) Plot the functions $c_n(x)$ and $\cos x$ in an (x, y) -diagram for the values $n = 1, 2, \dots, 10$. Choose as in (b) $x \in [-2\pi, +2\pi]$ and $y \in [-2, +2]$.

color code exercises:	
red exercises	necessary to pass exam
blue exercises	only for grade very good
green exercises	additional information, not relevant for exam
black exercises	only for R-Test