

**1. Exercise Sheet Advanced Mathematics
for Applied Physics and Biomedical Engineering**

Exercise 1) Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Calculate all real or complex eigenvalues of A and B and their corresponding eigenvectors.

Exercise 2) Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

- a) Calculate all eigenvalues of A .
- b) Determine an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors.
- c) Let V be the matrix

$$V = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$$

where $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the orthonormal basis of eigenvectors determined in part (b).
Check through explicit calculation:

$$A = V D V^T$$

if D denotes the diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ on the diagonal.

Exercise 3) The rotation matrices

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

are complex diagonalizable, there is a unitary matrix $U \in \mathbb{C}^{2 \times 2}$ and complex eigenvalues λ_1 and λ_2 such that

$$A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^* \quad (1)$$

with adjoint matrix $U^* = \bar{U}^T$. Calculate U and both eigenvalues λ_1 and λ_2 and check the identity (1) through explicit calculation.

Exercise 4) The scalar product of two vectors $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ is defined through

$$(\vec{x}, \vec{y}) := x_1 y_1 + \dots + x_n y_n = \sum_{j=1}^n x_j y_j$$

Let $A \in \mathbb{R}^{n \times n}$ be an arbitrary $n \times n$ matrix. Prove the following identities:

a) We have

$$(\vec{A}\vec{x}, \vec{y}) = (\vec{x}, \vec{A}^T \vec{y}) \quad (2)$$

with

$$A = \left(\begin{array}{c|ccc|c} & & & & & \\ & & & & & \\ \hline & \vec{a}_1 & \cdots & \vec{a}_n & & \\ \hline & & & & & \end{array} \right), \quad A^T = \left(\begin{array}{ccc|c} - & \vec{a}_1 & - & \\ & \vdots & & \\ - & \vec{a}_n & - & \end{array} \right)$$

or $[A^T]_{i,j} = a_{j,i}$ if $A = (a_{i,j})_{i,j=1,\dots,n}$.

b) For arbitrary $n \times n$ matrices A and B we have

$$(\vec{A}\vec{B})^T = \vec{B}^T \vec{A}^T.$$