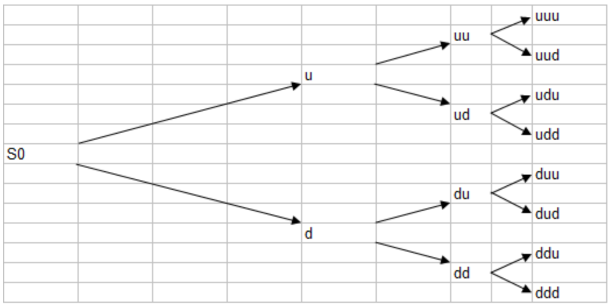
week6: Kapitel 5: Risikoneutrale Wahrscheinlichkeiten, Teil1

Mit dem Theorem 4.1 konnten wir die Preise von beliebigen pfadabhängigen oder pfadunabhängigen Optionen berechnen. Der Optionspreis V_0 war dabei durch eine rekursive Formel gegeben. Es ist nun möglich, auch eine direkte, nicht rekursive, Formel für V_0 anzugeben. Die sieht so aus:

$$V_0 \;=\; (1+r)^{-N}\, \mathsf{E}_{\rm rn}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{1}$$

wobei wir das $\mathsf{E}_{\rm rn}[\,\cdot\,]$ im folgenden noch genauer erklären werden. Insbesondere für pfadabhängige Optionen ist eine nicht-rekursive Pricing-Formel von besonderer Bedeutung, da wir ja letztes Mal gesehen haben, dass man bei pfadabhängigen Optionen einen nicht-rekombinierenden Binomialbaum der Form



betrachten muss. Dieser hat bei N Perioden 2^N Endpunkte und wenn man etwa $N = 250$ für eine Laufzeit von einem Jahr mit etwa 250 Handelstagen nimmt, dann bekommt man $2^{250} \approx 10^{75.26}$ Endpunkte, das ist nicht mehr berechenbar (die Anzahl der Atome im Universum wird auf 10^{80} geschätzt..). Also schauen wir uns jetzt an, wie man auf die direkte Formel (1) kommt.

Risk Neutral Probabilities

We go back to Definition 4.2, that was the definition of the Binomial model, and introduce some probabilities. That is, we write

$$S_k \;=\; S_{k-1} \times \begin{cases} (1+\mathrm{ret_{up}}) & \text{with some probability } \; p \\ (1+\mathrm{ret_{down}}) & \text{with probability } \; 1-p \end{cases} \tag{2}$$

thereby making the price process $\{S_k\}_{k=0}^N$ to a stochastic process. We know already that option prices do not depend on p. Now we use that fact to make a special choice for p which will allow us to calculate option prices, also for path dependent options, in a practical and efficient way. From Theorem 4.1, we know that payoff replication is possible. For zero interest rates, we have (the general case $r > 0$ will be considered below)

$$H(S_0, S_1, ..., S_N) \;=\; V_0 \;+\; \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (S_k - S_{k-1}) \qquad (3)$$

and V_0, the money which is needed to set up the replicating strategy, is the option price, the theoretical fair value of H. Let us introduce the following notation: For any function $f = f(S_0, S_1, ..., S_N)$ of the price process $\{S_k\}_{k=0}^N$ we introduce, for time t_k, a so called conditional expectation

$$\mathsf{E}\big[\, f(S_0, S_1, ..., S_N) \,|\, \{S_j\}_{j=0}^k \,\big] \qquad (4)$$

by considering the time point t_k as the actual present time such that $S_k, S_{k-1}, \ldots, S_0$ are actually known but the prices $S_{k+1}, S_{k+2}, ..., S_N$ are unknown since they are still in the future. That is, the $S_0, ..., S_k$ are deterministic quantities given by some realization of returns $\mathrm{ret}_{\mathrm{up}}$, $\mathrm{ret}_{\mathrm{down}}$, $\mathrm{ret}_{\mathrm{down}}$, ... , $\mathrm{ret}_{\mathrm{up}}$ (k returns have realized), but the $S_{k+1}, S_{k+2}, ..., S_N$ are still random, stochastic quantities since the future returns haven't realized yet. As an example, let us calculate the conditional expectation

$$\mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \qquad (5)$$

According to (2) we have

$$S_{k+1} \;=\; S_k \times (1 + \mathrm{ret}_{k+1}) \qquad (6)$$

with

$$\mathrm{ret}_{k+1} \;=\; \begin{cases} \mathrm{ret}_{\mathrm{up}} & \text{with probability } \; p \\ \mathrm{ret}_{\mathrm{down}} & \text{with probability } \; 1-p \end{cases} \qquad (7)$$

Thus,

$$\begin{aligned} \mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] &= \mathsf{E}\big[\, S_k \times (1 + \mathrm{ret}_{k+1}) \,|\, \{S_j\}_{j=0}^k \,\big] \\ &= S_k \times \mathsf{E}\big[\, 1 + \mathrm{ret}_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \\ &= S_k \times \Big(1 \,+\, \mathsf{E}\big[\mathrm{ret}_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big]\Big) \\ &= S_k \times \Big(1 \,+\, \mathrm{ret}_{\mathrm{up}} \cdot p \,+\, \mathrm{ret}_{\mathrm{down}} \cdot (1-p)\Big) \qquad (8) \end{aligned}$$

The choice of p which makes the conditional expectation (8) equal to S_k

$$\mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;\stackrel{!}{=}\; S_k \qquad (9)$$

is called the **risk neutral probability** (in case of zero interest rates). A stochastic process which fulfills equation (9) for all k is called a martingale. For zero interest rates, this risk neutral probability is obtained through

$$\begin{aligned} & S_k \times \big(1 \,+\, \mathrm{ret}_{\mathrm{up}} p + \mathrm{ret}_{\mathrm{down}}(1-p)\big) \;\stackrel{!}{=}\; S_k \\ \Leftrightarrow \quad & \mathrm{ret}_{\mathrm{up}} p + \mathrm{ret}_{\mathrm{down}}(1-p) \;=\; 0 \\ \Leftrightarrow \quad & (\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}})p \;=\; -\mathrm{ret}_{\mathrm{down}} \end{aligned}$$

which gives

$$p \;=\; \frac{-\mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}}-\mathrm{ret}_{\mathrm{down}}} \;=:\; p_{\text{risk neutral}} \tag{10}$$

Apparently the down return $\mathrm{ret}_{\mathrm{down}}$ has to be a negative number to obtain a meaningful p. Now let us fix p to this value (10) and to be more explicit we will use the notation $\mathsf{E} = \mathsf{E}_{\mathrm{rn}}$, 'rn' for 'risk neutral', to indicate that we are calculating expectation values using the risk neutral probability (10). The importance of this definition is due to the following important calculation:

$$\begin{aligned} V_0 \;&=\; \mathsf{E}_{\mathrm{rn}}[\,V_0\,] \;=\; \mathsf{E}_{\mathrm{rn}}[\,V_0 \,|\, S_0\,] \\ &=\; \mathsf{E}_{\mathrm{rn}}\Big[\,H(S_0,S_1,...,S_N) \;-\; \sum_{k=1}^{N}\delta_{k-1}(S_0,...,S_{k-1})\times(S_k-S_{k-1})\;\Big|\;S_0\,\Big] \\ &=\; \mathsf{E}_{\mathrm{rn}}\big[\,H(S_0,S_1,...,S_N)\,\big] \;-\; \sum_{k=1}^{N}\mathsf{E}_{\mathrm{rn}}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times(S_k-S_{k-1})\,\Big|\,S_0\,\Big] \end{aligned} \tag{11}$$

The expectations in the sum on the right hand side of (11) can be calculated as follows:

$$\begin{aligned} &\mathsf{E}_{\mathrm{rn}}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times(S_k-S_{k-1})\,\Big|\,S_0\,\Big] \;= \\ &=\; \mathsf{E}_{\mathrm{rn}}\Big[\,\underbrace{\mathsf{E}_{\mathrm{rn}}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times(S_k-S_{k-1})\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]}_{\text{in this expectation all } \mathrm{S}_1,...,\mathrm{S}_{k-1} \text{ are constant}}\;\Big|\;S_0\,\Big] \\ &=\; \mathsf{E}_{\mathrm{rn}}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times\mathsf{E}_{\mathrm{rn}}\Big[\,S_k-S_{k-1}\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]\;\Big|\;S_0\,\Big] \\ &=\; \mathsf{E}_{\mathrm{rn}}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times\Big(\mathsf{E}_{\mathrm{rn}}\Big[\,S_k\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big] \;-\; S_{k-1}\Big)\;\Big|\;S_0\,\Big] \end{aligned} \tag{12}$$

And now the decisive property of the risk neutral probability comes into play, namely:

$$\mathsf{E}_{\mathrm{rn}}\Big[\,S_k\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big] \;-\; S_{k-1} \;\overset{(9)}{=}\; S_{k-1} \;-\; S_{k-1} \;=\; 0 \tag{13}$$

Thus also the expectation (12) vanishes and therefore the whole sum on the right hand side of (11) goes away if we take an expectation with respect to the risk neutral probability. Hence we end up with the compact pricing formula

$$V_0 \;=\; \mathsf{E}_{\mathrm{rn}}\big[\,H(S_0,S_1,...,S_N)\,\big] \tag{14}$$

The argument generalizes to non zero interest rates and we summarize the result in the following theorem.

Theorem 5.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model (2). Let r be the interest rate paid per period and denote by

$$s_k \;=\; (1+r)^{-k}\,S_k \tag{15}$$

the discounted price process. Then the following statements hold:

a) Define the risk neutral probability

$$p_{\mathrm{rn}} \;=\; p_{\text{risk neutral}} \;\;:=\;\; \frac{r \,-\, \mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}} \,-\, \mathrm{ret}_{\mathrm{down}}} \tag{16}$$

and denote expectations with respect to this probability by $\mathsf{E}_{\mathrm{rn}}[\,\cdot\,]$. Then the discounted price process $\{s_k\}_{k=0}^N$ is a martingale with respect to the risk neutral expectation. That is, the following equation holds for all $k = 0, 1, 2, ..., N-1$:

$$\mathsf{E}_{\mathrm{rn}}\big[\, s_{k+1} \,|\, \{s_j\}_{j=0}^k \,\big] \;=\; s_k \tag{17}$$

b) Let $H = H(S_0, S_1, ..., S_N)$ be the payoff of some option. Then the theoretical fair value of this option can be obtained from the following risk neutral expectation:

$$V_0 \;=\; (1+r)^{-N}\, \mathsf{E}_{\mathrm{rn}}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{18}$$

Proof: The fact that option payoffs can be exactly replicated in the Binomial model reads in the presence of interest rates $r \neq 0$

$$h(S_0, S_1, ..., S_N) \;=\; v_0 \;+\; \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \tag{19}$$

with $h = (1+r)^{-N} H =: R^{-N} H$ being the discounted payoff function. Thus, if we want to eliminate the sum on the right hand side of (19) by taking an expectation value, we need to have the following property:

$$\mathsf{E}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;\stackrel{!}{=}\; s_k \tag{20}$$

or

$$R^{-(k+1)}\, \mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;\stackrel{!}{=}\; R^{-k}\, S_k \tag{21}$$

which is equivalent to

$$\begin{aligned} S_k \times \Big(1 \,+\, \mathrm{ret}_{\mathrm{up}} \cdot p \,+\, \mathrm{ret}_{\mathrm{down}} \cdot (1-p)\Big) \;&\stackrel{!}{=}\; R\, S_k \\ \Leftrightarrow \qquad \mathrm{ret}_{\mathrm{up}} \cdot p \,+\, \mathrm{ret}_{\mathrm{down}} \cdot (1-p) \;&=\; R - 1 \;=\; r \\ \Leftrightarrow \qquad (\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}) p \;&=\; r - \mathrm{ret}_{\mathrm{down}} \end{aligned}$$

which gives

$$p \;=\; \frac{r - \mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \;=:\; p_{\text{risk neutral}} \;=:\; p_{\mathrm{rn}}$$

This proves part (a). Part (b) is obtained in the same way as above: we can write

$$\begin{aligned} V_0 \;= v_0 \;\;&=\;\; \mathsf{E}_{\mathrm{rn}}[\,v_0\,] \;=\; \mathsf{E}_{\mathrm{rn}}[\,v_0 \,|\, S_0\,] \\ &\stackrel{(19)}{=}\; \mathsf{E}_{\mathrm{rn}}\Big[\, h(S_0, S_1, ..., S_N) \;-\; \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \,\Big|\, S_0 \Big] \\ &=\;\; \mathsf{E}_{\mathrm{rn}}\big[\, h(S_0, S_1, ..., S_N)\,\big] \;-\; \sum_{k=1}^{N} \mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \,\Big|\, S_0 \Big] \end{aligned} \tag{22}$$

The expectations in the sum on the right hand side of (22) can be calculated as follows:

$$\begin{aligned}
&\mathsf{E}_{\rm rn}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times(s_k-s_{k-1})\,\Big|\,S_0\,\Big] \;= \\
&\quad=\;\mathsf{E}_{\rm rn}\Bigg[\,\underbrace{\mathsf{E}_{\rm rn}\Big[\,\delta_{k-1}(S_0,...,S_{k-1})\times(s_k-s_{k-1})\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]}_{\text{in this expectation all } \mathrm{S_1,...,S_{k-1}} \text{ are constant}}\,\Bigg|\,S_0\,\Bigg] \\
&\quad=\;\mathsf{E}_{\rm rn}\Bigg[\,\delta_{k-1}(S_0,...,S_{k-1})\times\mathsf{E}_{\rm rn}\Big[\,s_k-s_{k-1}\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]\,\Bigg|\,S_0\,\Bigg] \\
&\quad=\;\mathsf{E}_{\rm rn}\Bigg[\,\delta_{k-1}(S_0,...,S_{k-1})\times\Big(\mathsf{E}_{\rm rn}\Big[\,s_k\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]\;-\;s_{k-1}\Big)\,\Bigg|\,S_0\,\Bigg] \qquad (23)
\end{aligned}$$

Now we use the martingale property (20)

$$\mathsf{E}_{\rm rn}\Big[\,s_k\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big]\;-\;s_{k-1}\;=\;s_{k-1}\;-\;s_{k-1}\;=\;0 \qquad (24)$$

Thus also the expectation (12) vanishes and therefore the whole sum on the right hand side of (11) goes away if we take an expectation with respect to the risk neutral probability. Hence we end up with

$$\begin{aligned}
V_0 \;=\; v_0 \;&=\; \mathsf{E}_{\rm rn}\big[\,h(S_0,S_1,...,S_N)\,\big] \\
&=\; \mathsf{E}_{\rm rn}\big[\,R^{-N}\,H(S_0,S_1,...,S_N)\,\big] \\
&=\; R^{-N}\,\mathsf{E}_{\rm rn}\big[\,H(S_0,S_1,...,S_N)\,\big]
\end{aligned}$$

and the theorem is proven. ■