week4: Kapitel 4: Das N-Perioden Binomialmodell, Teil1

Let S be some tradable asset with prices

$$S_k \;=\; S(t_k), \quad k = 0, 1, 2, \dots \tag{1}$$

and let

$$H \;=\; H(S_0, S_1, \dots, S_{N-1}, S_N) \tag{2}$$

be some option payoff with start date t_0 and end date or maturity t_N. We want to replicate the option payoff (2) with a suitable trading strategy in the underlying S. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding δ_k assets at the end of day t_k generates the amount

$$V_N \;=\; V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{3}$$

Each δ_k will be determined on the end of trading day t_k. On such a day, the asset prices $S_0, S_1, \dots, S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, \dots, S_N$ are not known yet, they are lying in the future. Thus, δ_k can be a function only of the known prices $S_0, \dots, S_k$,

$$\delta_k \;=\; \delta_k(S_0, S_1, \dots, S_{k-1}, S_k) \tag{4}$$

Definition 4.1: We say that an option payoff (2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of δ_k of the form (4) and some initial amount V_0 such that (in case of zero interest rates)

$$H(S_0, S_1, \dots, S_{N-1}, S_N) \;=\; V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{5}$$

The **initial amount V_0** which is needed to set up the replicating strategy is called the theoretical fair value of H or **the price of the option H.** The process of replicating an option payoff H through formula (5), that is, through a trading strategy which holds δ_j pieces of the underlying S at the end of day t_j, is called **hedging**.

Now let us consider the question to what extent replication of options is possible. Equation (5) can be rewritten as

$$H(S_0, S_1, \dots, S_{N-1}, S_N) \;=\; V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$H(S_0, S_1, ..., S_{N-1}, S_N) - \delta_{N-1} S_N = V_{N-1} - \delta_{N-1} S_{N-1}$$
$$= \text{some function of } S_0, S_1, ..., S_{N-1} \qquad (6)$$

That is, the right hand side of (6) is independent of S_N. Let us introduce the return of the asset S from t_{k-1} to t_k,

$$\text{ret}_k := \frac{S_k - S_{k-1}}{S_{k-1}} = \frac{S_k}{S_{k-1}} - 1 \qquad (7)$$

such that

$$S_k = S_{k-1}(1 + \text{ret}_k) \qquad (8)$$

Then equation (6) can be rewritten as

$$H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_N)\big) - \delta_{N-1} S_{N-1}(1 + \text{ret}_N) = \text{const} \qquad (9)$$

where the const in the equation above means that the left hand side of (9) has to be the same for all possible choices of ret_N. Since there is only 1 free parameter in (9), namely δ_{N-1}, we can only allow for 2 possible choices for ret_N, say,

$$\text{ret}_N \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \qquad (10)$$

and in that case we have to have

$$H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})\big) - \delta_{N-1} S_{N-1}(1 + \text{ret}_{\text{up}}) =$$
$$H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}})\big) - \delta_{N-1} S_{N-1}(1 + \text{ret}_{\text{down}})$$

which determines δ_{N-1} to

$$\delta_{N-1} = \frac{H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})\big) - H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}})\big)}{S_{N-1}(1 + \text{ret}_{\text{up}}) - S_{N-1}(1 + \text{ret}_{\text{down}})} \qquad (11)$$

Thus, if we allow for 2 possible choices of returns only as in (10), replication of option payoffs seem to be possible. This leads to the following

Definition 4.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$S_k = S_{k-1}(1 + \text{ret}_k) \quad \text{with} \quad \text{ret}_k \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \qquad (12)$$

for all k, then we say that S is given by the Binomial model.

Remark: Observe that in Definition 4.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability p_{up} such that an up-return ret_{up} will occur and a probability $p_{\text{down}} = 1 - p_{\text{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value V_0 are actually independent of such probabilities.

Now we are in a position to formulate the following important

Theorem 4.1: Let S be some tradable asset whose price process is given by the Binomial model (12). Let r be some interest rate per period such that cash amounts G change their values according to $G \stackrel{t_{k-1} \to t_k}{\to} G(1+r)$. Then every option payoff

$$H = H(S_0, ..., S_N)$$

can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k = \delta_k(S_0, \cdots, S_k) = \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} \tag{13}$$

with the abbreviations

$$S_{k+1}^{\text{up/down}} := S_k \left(1 + \text{ret}_{\text{up/down}} \right)$$

$$V_{k+1}^{\text{up/down}} := V_{k+1} \left(S_0, \cdots, S_k, S_{k+1}^{\text{up/down}} \right)$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae:

$$V_k = (1+r)^k v_k$$

with discounted portfolio values v_k given recursively by

$$v_k = w_{\text{up}} v_{k+1}^{\text{up}} + w_{\text{down}} v_{k+1}^{\text{down}} \tag{14}$$

and the recursion starts at $k = N$ with discounted portfolio values

$$v_N := (1+r)^{-N} H(S_0, \cdots, S_N)$$

The weights w_{up} and w_{down} are given by

$$w_{\text{up}} = \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \tag{15}$$

$$w_{\text{down}} = 1 - w_{\text{up}} = \frac{\text{ret}_{\text{up}} - r}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \tag{16}$$

Proof: For nonzero interest rates we have

$$v_N = v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$

which is equivalent to

$$v_{k+1} = v_k + \delta_k(s_{k+1} - s_k) \qquad \forall \, k = 0, 1, ..., N-1 \tag{17}$$

We have

$$v_{k+1} \;=\; v_{k+1}(S_0,\cdots,S_k,S_{k+1}) \;=\; v_{k+1}\big(S_0\,,\,\cdots,\,S_k\,,\,S_k(1+\mathrm{ret}_k)\big)$$

and the return ret_k can be an up-move or a down-move in which case we get

$$
\begin{aligned}
v_{k+1}^{\mathrm{up}} &= v_{k+1}\big(S_0\,,\,\cdots,\,S_k\,,\,S_{k+1}^{\mathrm{up}}\big) = v_{k+1}\big(S_0\,,\,\cdots,\,S_k\,,\,S_k(1+\mathrm{ret}_{\mathrm{up}})\big)\\
v_{k+1}^{\mathrm{down}} &= v_{k+1}\big(S_0\,,\,\cdots,\,S_k\,,\,S_{k+1}^{\mathrm{down}}\big) = v_{k+1}\big(S_0\,,\,\cdots,\,S_k\,,\,S_k(1+\mathrm{ret}_{\mathrm{down}})\big)
\end{aligned}
$$

From (17), we have

$$
\begin{aligned}
v_{k+1}^{\mathrm{up}} &= v_k + \delta_k(s_{k+1}^{\mathrm{up}} - s_k)\\
v_{k+1}^{\mathrm{down}} &= v_k + \delta_k(s_{k+1}^{\mathrm{down}} - s_k)
\end{aligned}
$$

Thus,

$$v_{k+1}^{\mathrm{up}} - v_{k+1}^{\mathrm{down}} = \delta_k(s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}})$$

or

$$\delta_k = \frac{v_{k+1}^{\mathrm{up}} - v_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}} = \frac{e^{-rt_{k+1}}(V_{k+1}^{\mathrm{up}} - V_{k+1}^{\mathrm{down}})}{e^{-rt_{k+1}}(S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}})} = \frac{V_{k+1}^{\mathrm{up}} - V_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}}}$$

Solving (17) for v_k,

$$
\begin{aligned}
v_k &= v_{k+1} - \delta_k(s_{k+1} - s_k)\\
&= v_{k+1}^{\mathrm{up}} - \delta_k(s_{k+1}^{\mathrm{up}} - s_k)\\
&= v_{k+1}^{\mathrm{down}} - \delta_k(s_{k+1}^{\mathrm{down}} - s_k)
\end{aligned}
$$

Let's take the up-equation and substitute the value for δ_k (we also could use the down-equation, we would end up with the same result),

$$
\begin{aligned}
v_k &= v_{k+1}^{\mathrm{up}} - \delta_k(s_{k+1}^{\mathrm{up}} - s_k)\\[2mm]
&= v_{k+1}^{\mathrm{up}} - \frac{v_{k+1}^{\mathrm{up}} - v_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,(s_{k+1}^{\mathrm{up}} - s_k)\\[2mm]
&= \frac{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,v_{k+1}^{\mathrm{up}} - (v_{k+1}^{\mathrm{up}} - v_{k+1}^{\mathrm{down}})\frac{s_{k+1}^{\mathrm{up}} - s_k}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\\[2mm]
&= \frac{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}} - s_{k+1}^{\mathrm{up}} + s_k}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,v_{k+1}^{\mathrm{up}} + \frac{s_{k+1}^{\mathrm{up}} - s_k}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,v_{k+1}^{\mathrm{down}}\\[2mm]
&= \frac{s_k - s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,v_{k+1}^{\mathrm{up}} + \frac{s_{k+1}^{\mathrm{up}} - s_k}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}}\,v_{k+1}^{\mathrm{down}}\\[2mm]
&=: w_{\mathrm{up}}\,v_{k+1}^{\mathrm{up}} + w_{\mathrm{down}}\,v_{k+1}^{\mathrm{down}}
\end{aligned}
$$

with weights w_{up} and w_{down} which apparently add up to 1 and, using the abbreviation $R := 1 + r$,

$$w_{\mathrm{up}} = \frac{s_k - s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} - s_{k+1}^{\mathrm{down}}} = \frac{R^{-k}S_k - R^{-(k+1)}S_{k+1}^{\mathrm{down}}}{R^{-(k+1)}(S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}})} = \frac{RS_k - S_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}}}$$

or

$$w_{\text{up}} = \frac{(1+r)S_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} = \frac{(1+r)S_k - S_k(1+\text{ret}_{\text{down}})}{S_k(1+\text{ret}_{\text{up}}) - S_k(1+\text{ret}_{\text{down}})}$$

$$= \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}$$

and the theorem is proven. ∎

Beispiel: Standard-Kauf-Option in einem 3-Perioden Binomialmodell → Ü-Blatt 4