

week11: Kapitel 7: Die Black-Scholes Formeln, Teil2

Wir benutzen jetzt die grundlegende Pricing-Formel aus dem Theorem 7.1 vom letzten Mal,

$$V_0^{\text{BS}} = e^{-rT} \int_{\mathbb{R}} H\left(S_0 e^{\sigma\sqrt{T}x + (r - \frac{\sigma^2}{2})T}\right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad (1)$$

um die Black-Scholes Formeln herzuleiten. Dann definieren und berechnen wir noch ein paar ‘Greeks’, das sind einfach Ableitungen des Optionspreises nach den verschiedenen Parametern wie S_t , σ , r oder t . Bevor wir das machen, schreiben wir eben noch eine leichte Verallgemeinerung von (1) hin, nämlich den Black-Scholes Preis einer pfadunabhängigen Option $H = H(S_T)$ nicht zur Zeit 0, sondern an einem beliebigen Zeitpunkt $t \in [0, T]$, das grosse T bezeichnet dabei wieder die Laufzeit der Option H . Mit denselben Überlegungen wie im week10.pdf bekommt man die folgende Formel:

Folgerung 7.1: Es sei $H = H(S_T)$ die Auszahlungsfunktion einer beliebigen, pfadunabhängigen Option. Dann ist der Zeit- t Black-Scholes Preis dieser Option gegeben durch das eindimensionale Integral

$$V_t^{\text{BS}} = e^{-r(T-t)} \int_{\mathbb{R}} H\left(S_t e^{\sigma\sqrt{T-t}x + (r - \frac{\sigma^2}{2})(T-t)}\right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}. \quad (2)$$

Der Optionspreis $V_t = V_t^{\text{BS}}$, im Folgenden wollen wir den Superscript ‘BS’ weglassen, ist also eine Funktion von den folgenden Parametern:

$$V_t = V(S_t, \sigma, r, t, T) \quad (3)$$

Was die Abhängigkeit von den Zeit-Parametern angeht, könnte man auch schreiben $V = V(T - t)$, aber das θ in der Definition 7.1 gleich weiter unten ist die partielle Ableitung von (3) nach t , nicht die Ableitung von V nach der Restlaufzeit $\tau = T - t$, das unterscheidet sich dann um ein Vorzeichen. Die Options-Sensitivitäten oder kurz ‘Greeks’, (weil diese Größen typischerweise mit griechischen Buchstaben bezeichnet werden) sind dann wie folgt definiert:

Definition 7.1: Let $H = H(S_T)$ be some option payoff and let

$$V_t = V(S_t, \sigma, r, t, T) \quad (4)$$

be the option price at time t given by formula (2) above. Here S_t is the asset price at time t , σ denotes the volatility and r denotes the interest rates. Then the following abbreviations are standard in the derivatives community (listed with decreasing importance):

$$\delta := \frac{\partial V_t}{\partial S_t} \quad (\text{delta}) \quad (5)$$

$$vega := \frac{\partial V_t}{\partial \sigma} \quad (\text{vega}) \quad (6)$$

$$\rho := \frac{\partial V_t}{\partial r} \quad (\text{rho}) \quad (7)$$

$$\theta := \frac{\partial V_t}{\partial t} \quad (\text{theta}) \quad (8)$$

$$\gamma := \frac{\partial^2 V_t}{\partial S_t^2} \quad (\text{gamma}) \quad (9)$$

Theorem 7.2 (Black-Scholes Formeln): Consider a standard call and put option with strike K , maturity T and payoffs

$$H_{\text{call}} = \max\{S_T - K, 0\} \quad (10)$$

$$H_{\text{put}} = \max\{K - S_T, 0\} \quad (11)$$

and let $V_{\text{call},t}$ and $V_{\text{put},t}$ denote their time t theoretical fair values in the Black-Scholes model. Let

$$N(x) := \int_{-\infty}^x e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \quad (12)$$

denote the cumulative normal distribution function and define the quantities

$$d_{\pm} := \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \quad (13)$$

such that $d_+ - d_- = \sigma \sqrt{T - t}$. Then there are the following formulae:

Option Price:

$$V_{\text{call},t} = S_t N(d_+) - K e^{-r(T-t)} N(d_-) \quad (14)$$

$$V_{\text{put},t} = -S_t N(-d_+) + K e^{-r(T-t)} N(-d_-) \quad (15)$$

Delta:

$$\delta_{\text{call}} = N(d_+) \quad (16)$$

$$\delta_{\text{put}} = -N(-d_+) \quad (17)$$

Vega:

$$vega_{\text{call}} = S_t N'(d_+) \sqrt{T - t} \quad (18)$$

$$vega_{\text{put}} = vega_{\text{call}} \quad (19)$$

Rho:

$$\rho_{\text{call}} = K (T - t) e^{-r(T-t)} N(d_-) \quad (20)$$

$$\rho_{\text{put}} = -K (T - t) e^{-r(T-t)} N(-d_-) \quad (21)$$

Theta:

$$\theta_{\text{call}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(d_+) - r K e^{-r(T-t)} N(d_-) \quad (22)$$

$$\theta_{\text{put}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(-d_+) + r K e^{-r(T-t)} N(-d_-) \quad (23)$$

Gamma:

$$\gamma_{\text{call}} = \frac{N'(d_+)}{\sigma\sqrt{T-t}S_t} \quad (24)$$

$$\gamma_{\text{put}} = \gamma_{\text{call}} \quad (25)$$

Proof: In the Black-Scholes model, the time t theoretical fair value of some european option with maturity $T \geq t$ and payoff $H = H(S_T)$ is given by

$$V_t = e^{-r(T-t)} \int_{\mathbb{R}} H\left(S_t e^{\sigma\sqrt{T-t}x + (r-\frac{\sigma^2}{2})(T-t)}\right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad (26)$$

Let us abbreviate

$$\tau := T - t \quad (27)$$

$$a := \sigma\sqrt{\tau} \quad (28)$$

$$b := (r - \sigma^2/2)\tau \quad (29)$$

and define x_0 as the solution of the equation

$$S_t e^{\sigma\sqrt{T-t}x + (r-\frac{\sigma^2}{2})(T-t)} = S_t e^{ax+b} \stackrel{!}{=} K \quad (30)$$

That is,

$$x_0 = \frac{1}{a} \log(K/S_t) - \frac{b}{a} \quad (31)$$

Then

$$\begin{aligned} V_{\text{call},t} &= e^{-r\tau} \int_{\mathbb{R}} H_{\text{call}}(S_t e^{ax+b}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{\mathbb{R}} \max\{S_t e^{ax+b} - K, 0\} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} S_t e^{b+\frac{a^2}{2}} \int_{x_0}^{\infty} e^{-\frac{1}{2}(x-a)^2} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} K \int_{x_0}^{\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} S_t e^{b+\frac{a^2}{2}} N(-x_0 + a) - e^{-r\tau} K N(-x_0) \\ &= S_t N(d_+) - e^{-r(T-t)} K N(d_-) \end{aligned} \quad (32)$$

since

$$-x_0 = \frac{1}{a} \log(S_t/K) + \frac{b}{a} = \frac{\log(S_t/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_-$$

and $-x_0 + a = d_- + \sigma\sqrt{\tau} = d_+$. Furthermore

$$-r\tau + b + a^2/2 = -r\tau + (r - \sigma^2/2)\tau + (\sigma\sqrt{\tau})^2/2 = 0$$

which results in the last line of (32). The proof for the put fair value is similar. In particular, since

$$N(d) + N(-d) = 1$$

we get from (14,15)

$$V_{\text{call},t} - V_{\text{put},t} = S_t - e^{-r(T-t)}K \quad (33)$$

This relation is called ‘call-put parity’.

Delta: To compute the delta we use the third line of (32) instead of the last one. We get

$$\begin{aligned} \delta_{\text{call}}(S_t, t) &= \frac{\partial}{\partial S_t} \left\{ e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right\} \\ &= e^{-r\tau} \int_{x_0}^{\infty} e^{ax+b} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} \underbrace{(S_t e^{ax_0+b} - K)}_{=0} \frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}} \frac{\partial x_0}{\partial S_t} \\ &= N(d_+) \end{aligned} \quad (34)$$

since the above integral is the same as the first integral in (32), up to the factor of S_t which has been differentiated away. Finally it follows from call-put parity that

$$\delta_{\text{put}}(S, t) = N(d_+) - 1 = N(-d_+).$$

Vega: Again we use the third line of (32):

$$\begin{aligned} \text{vega}_{\text{call}}(S_t, t) &= \frac{\partial}{\partial \sigma} \left\{ e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right\} \\ &= e^{-r\tau} \int_{x_0}^{\infty} S_t \left(\frac{\partial a}{\partial \sigma} x + \frac{\partial b}{\partial \sigma} \right) e^{ax+b} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} \underbrace{(S_t e^{ax_0+b} - K)}_{=0} \frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}} \frac{\partial x_0}{\partial \sigma} \\ &= e^{-r\tau} e^{b + \frac{a^2}{2}} S_t \int_{x_0}^{\infty} (\sqrt{\tau}x - \sigma\tau) e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= S_t \sqrt{\tau} \int_{x_0}^{\infty} (x - a) e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= S_t \sqrt{\tau} \left[-\frac{e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}} \right]_{x_0}^{\infty} = S_t \sqrt{\tau} \frac{e^{-\frac{(x_0-a)^2}{2}}}{\sqrt{2\pi}} \\ &= S_t \sqrt{\tau} \frac{e^{-\frac{(-d_+)^2}{2}}}{\sqrt{2\pi}} = S_t \sqrt{\tau} \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} = S_t \sqrt{\tau} N'(d_+) \end{aligned} \quad (35)$$

The vega for the put follows again from call-put parity. The remaining formulae may be looked up, for example, in the book on Quantitative Finance, Vol.1, by Paul Wilmott. ■