

Lösungen zum 11. Übungsblatt Dynamik der Teilchen und Felder

2.Aufgabe: Mit der mehrdimensionalen Kettenregel bekommen wir

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial \varepsilon} &= \frac{\partial H}{\partial x} \frac{\partial x}{\partial \varepsilon} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi} &= \frac{\partial H}{\partial x} \frac{\partial x}{\partial \varphi} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \varphi}\end{aligned}$$

oder, wenn wir die Hamiltonschen Gleichungen

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

benutzen,

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial \varepsilon} &= -\frac{\partial x}{\partial \varepsilon} \dot{p} + \frac{\partial p}{\partial \varepsilon} \dot{x} \\ \frac{\partial \tilde{H}}{\partial \varphi} &= -\frac{\partial x}{\partial \varphi} \dot{p} + \frac{\partial p}{\partial \varphi} \dot{x}\end{aligned}$$

In Matrix-Notation,

$$\begin{pmatrix} \frac{\partial \tilde{H}}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix}$$

Weiterhin,

$$\begin{aligned}\dot{x} &= \frac{\partial x}{\partial \varepsilon} \dot{\varepsilon} + \frac{\partial x}{\partial \varphi} \dot{\varphi} \\ \dot{p} &= \frac{\partial p}{\partial \varepsilon} \dot{\varepsilon} + \frac{\partial p}{\partial \varphi} \dot{\varphi}\end{aligned}$$

oder in Matrix-Notation,

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \dot{\varepsilon} \\ \dot{\varphi} \end{pmatrix}$$

Also,

$$\begin{pmatrix} \frac{\partial \tilde{H}}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \dot{\varepsilon} \\ \dot{\varphi} \end{pmatrix} =: J \begin{pmatrix} \dot{\varepsilon} \\ \dot{\varphi} \end{pmatrix} \quad (1)$$

mit der Matrix

$$\begin{aligned}J &:= \begin{pmatrix} \frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{\partial p}{\partial \varepsilon} \frac{\partial x}{\partial \varphi} - \frac{\partial x}{\partial \varepsilon} \frac{\partial p}{\partial \varphi} \\ \frac{\partial p}{\partial \varphi} \frac{\partial x}{\partial \varepsilon} - \frac{\partial x}{\partial \varphi} \frac{\partial p}{\partial \varepsilon} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & -j(\varepsilon, \varphi) \\ j(\varepsilon, \varphi) & 0 \end{pmatrix} \quad (2)\end{aligned}$$

mit der skalaren Funktion

$$j(\varepsilon, \varphi) \quad := \quad \frac{\partial x}{\partial \varepsilon} \frac{\partial p}{\partial \varphi} - \frac{\partial x}{\partial \varphi} \frac{\partial p}{\partial \varepsilon}$$

Wir haben

$$\begin{aligned} \frac{\partial x}{\partial \varepsilon} &= \frac{1}{\sqrt{2\varepsilon}} \cos \varphi \\ \frac{\partial x}{\partial \varphi} &= -\sqrt{2\varepsilon} \sin \varphi \\ \frac{\partial p}{\partial \varepsilon} &= \frac{1}{\sqrt{2\varepsilon}} \sin \varphi \\ \frac{\partial p}{\partial \varphi} &= \sqrt{2\varepsilon} \cos \varphi \end{aligned}$$

und bekommen somit

$$j(\varepsilon, \varphi) \quad = \quad \frac{\partial x}{\partial \varepsilon} \frac{\partial p}{\partial \varphi} - \frac{\partial x}{\partial \varphi} \frac{\partial p}{\partial \varepsilon} \quad = \quad \cos^2 \varphi + \sin^2 \varphi \quad = \quad 1 \, .$$

Setzt man das in die Gleichungen (2) und (1) ein, folgt die Behauptung.