week4: Kapitel 4: Das N-Perioden Binomialmodell, Teil1

Let S be some tradable asset with prices

$$S_k \;=\; S(t_k), \quad k = 0, 1, 2, ... \tag{1}$$

and let

$$H \;=\; H(S_0, S_1, ..., S_{N-1}, S_N) \tag{2}$$

be some option payoff with start date t_0 and end date or maturity t_N. We want to replicate the option payoff (2) with a suitable trading strategy in the underlying S. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding δ_k assets at the end of day t_k generates the amount

$$V_N \;=\; V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{3}$$

Each δ_k will be determined on the end of trading day t_k. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, δ_k can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k \;=\; \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \tag{4}$$

Definition 4.1: We say that an option payoff (2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of δ_k of the form (4) and some initial amount V_0 such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_0 \;+\; \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{5}$$

The **initial amount $\mathbf{V_0}$** which is needed to set up the replicating strategy is called the theoretical fair value of H or **the price of the option H.** The process of replicating an option payoff H through formula (5), that is, through a trading strategy which holds δ_j pieces of the underlying S at the end of day t_j, is called **hedging**.

Now let us consider the question to what extent replication of options is possible. Equation (5) can be rewritten as

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$\begin{aligned} H(S_0,S_1,...,S_{N-1},S_N) - \delta_{N-1}\,S_N &= V_{N-1} - \delta_{N-1}\,S_{N-1} \\ &= \text{some function of } \; S_0,S_1,...,S_{N-1} \end{aligned} \tag{6}$$

That is, the right hand side of (6) is independent of S_N. Let us introduce the return of the asset S from t_{k-1} to t_k,

$$\mathrm{ret}_k \;:=\; \frac{S_k - S_{k-1}}{S_{k-1}} \;=\; \frac{S_k}{S_{k-1}} \,-\, 1 \tag{7}$$

such that

$$S_k \;=\; S_{k-1}(1+\mathrm{ret}_k) \tag{8}$$

Then equation (6) can be rewritten as

$$H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\mathrm{ret}_N)\big) - \delta_{N-1}\,S_{N-1}(1+\mathrm{ret}_N) \;=\; \text{const} \tag{9}$$

where the const in the equation above means that the left hand side of (9) has to be the same for all possible choices of ret_N. Since there is only 1 free parameter in (9), namely δ_{N-1}, we can only allow for 2 possible choices for ret_N, say,

$$\mathrm{ret}_N \;\in \{\mathrm{ret}_{\mathrm{up}},\mathrm{ret}_{\mathrm{down}}\} \tag{10}$$

and in that case we have to have

$$\begin{aligned} &H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\mathrm{ret}_{\mathrm{up}})\big) - \delta_{N-1}\,S_{N-1}(1+\mathrm{ret}_{\mathrm{up}}) \;= \\ &H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\mathrm{ret}_{\mathrm{down}})\big) - \delta_{N-1}\,S_{N-1}(1+\mathrm{ret}_{\mathrm{down}}) \end{aligned}$$

which determines δ_{N-1} to

$$\delta_{N-1} \;=\; \frac{H\big(S_0,...,S_{N-1},S_{N-1}(1+\mathrm{ret}_{\mathrm{up}})\big) \;-\; H\big(S_0,...,S_{N-1},S_{N-1}(1+\mathrm{ret}_{\mathrm{down}})\big)}{S_{N-1}(1+\mathrm{ret}_{\mathrm{up}}) \;-\; S_{N-1}(1+\mathrm{ret}_{\mathrm{down}})} \tag{11}$$

Thus, if we allow for 2 possible choices of returns only as in (10), replication of option payoffs seem to be possible. This leads to the following

Definition 4.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$S_k \;=\; S_{k-1}(\,1 \,+\, \mathrm{ret}_k\,) \qquad \text{with} \quad \mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}},\mathrm{ret}_{\mathrm{down}}\} \tag{12}$$

for all k, then we say that S is given by the Binomial model.

Remark: Observe that in Definition 4.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability p_{up} such that an up-return $\mathrm{ret}_{\mathrm{up}}$ will occur and a probability $p_{\mathrm{down}} = 1 - p_{\mathrm{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value V_0 are actually independent of such probabilities.

Now we are in a position to formulate the following important

Theorem 4.1: Let S be some tradable asset whose price process is given by the Binomial model (12). Let r be some interest rate per period such that cash amounts G change their values according to $G \overset{t_{k-1}\to t_k}{\to} G(1+r)$. Then every option payoff

$$H \;=\; H(S_0, ..., S_N)$$

can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \delta_k(S_0, \cdots, S_k) \;=\; \frac{V_{k+1}^{\mathrm{up}} - V_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}}} \tag{13}$$

with the abbreviations

$$S_{k+1}^{\mathrm{up/down}} \;:=\; S_k\,(\,1 \,+\, \mathrm{ret}_{\mathrm{up/down}}\,)$$
$$V_{k+1}^{\mathrm{up/down}} \;:=\; V_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\mathrm{up/down}}\,\big)$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae:

$$V_k \;=\; (1+r)^k\, v_k$$

with discounted portfolio values v_k given recursively by

$$v_k \;=\; w_{\mathrm{up}}\, v_{k+1}^{\mathrm{up}} \;+\; w_{\mathrm{down}}\, v_{k+1}^{\mathrm{down}} \tag{14}$$

and the recursion starts at $k = N$ with discounted portfolio values

$$v_N \;:=\; (1+r)^{-N}\, H(S_0, \cdots, S_N)$$

The weights w_{up} and w_{down} are given by

$$w_{\mathrm{up}} \;=\; \frac{r \,-\, \mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}} \,-\, \mathrm{ret}_{\mathrm{down}}} \tag{15}$$

$$w_{\mathrm{down}} \;=\; 1 - w_{\mathrm{up}} \;=\; \frac{\mathrm{ret}_{\mathrm{up}} \,-\, r}{\mathrm{ret}_{\mathrm{up}} \,-\, \mathrm{ret}_{\mathrm{down}}} \tag{16}$$

Proof: For nonzero interest rates we have

$$v_N \;=\; v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$

which is equivalent to

$$v_{k+1} \;=\; v_k + \delta_k(s_{k+1} - s_k) \qquad \forall\; k = 0, 1, ..., N-1 \tag{17}$$

We have

$$v_{k+1} \;=\; v_{k+1}(S_0,\cdots,S_k,S_{k+1}) \;=\; v_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\mathrm{ret}_k)\,\big)$$

and the return ret_k can be an up-move or a down-move in which case we get

$$\begin{aligned} v_{k+1}^{\mathrm{up}} &= v_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\mathrm{up}}\,\big) \;=\; v_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\mathrm{ret}_{\mathrm{up}})\,\big) \\ v_{k+1}^{\mathrm{down}} &= v_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\mathrm{down}}\,\big) \;=\; v_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_k(1+\mathrm{ret}_{\mathrm{down}})\,\big) \end{aligned}$$

From (17), we have

$$\begin{aligned} v_{k+1}^{\mathrm{up}} &= v_k + \delta_k(s_{k+1}^{\mathrm{up}} - s_k) \\ v_{k+1}^{\mathrm{down}} &= v_k + \delta_k(s_{k+1}^{\mathrm{down}} - s_k) \end{aligned}$$

Thus,

$$v_{k+1}^{\mathrm{up}} \;-\; v_{k+1}^{\mathrm{down}} \;=\; \delta_k(s_{k+1}^{\mathrm{up}} \;-\; s_{k+1}^{\mathrm{down}})$$

or

$$\delta_k \;=\; \frac{v_{k+1}^{\mathrm{up}} \;-\; v_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} \;-\; s_{k+1}^{\mathrm{down}}} \;=\; \frac{e^{-rt_{k+1}}(V_{k+1}^{\mathrm{up}} \;-\; V_{k+1}^{\mathrm{down}})}{e^{-rt_{k+1}}(S_{k+1}^{\mathrm{up}} \;-\; S_{k+1}^{\mathrm{down}})} \;=\; \frac{V_{k+1}^{\mathrm{up}} \;-\; V_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} \;-\; S_{k+1}^{\mathrm{down}}}$$

Solving (17) for v_k,

$$\begin{aligned} v_k &= v_{k+1} \;-\; \delta_k(s_{k+1} - s_k) \\ &= v_{k+1}^{\mathrm{up}} \;-\; \delta_k(s_{k+1}^{\mathrm{up}} - s_k) \\ &= v_{k+1}^{\mathrm{down}} \;-\; \delta_k(s_{k+1}^{\mathrm{down}} - s_k) \end{aligned}$$

Let's take the up-equation and substitute the value for δ_k (we also could use the down-equation, we would end up with the same result),

$$\begin{aligned} v_k &= v_{k+1}^{\mathrm{up}} \;-\; \delta_k(s_{k+1}^{\mathrm{up}} - s_k) \\ &= v_{k+1}^{\mathrm{up}} \;-\; \tfrac{v_{k+1}^{\mathrm{up}} \,-\, v_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\,(s_{k+1}^{\mathrm{up}} - s_k) \\ &= \tfrac{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\, v_{k+1}^{\mathrm{up}} \;-\; (v_{k+1}^{\mathrm{up}} \;-\; v_{k+1}^{\mathrm{down}})\, \tfrac{s_{k+1}^{\mathrm{up}} \,-\, s_k}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}} \\ &= \tfrac{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}} \,-\, s_{k+1}^{\mathrm{up}} \,+\, s_k}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\; v_{k+1}^{\mathrm{up}} \;+\; \tfrac{s_{k+1}^{\mathrm{up}} \,-\, s_k}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\; v_{k+1}^{\mathrm{down}} \\ &= \tfrac{s_k \,-\, s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\; v_{k+1}^{\mathrm{up}} \;+\; \tfrac{s_{k+1}^{\mathrm{up}} \,-\, s_k}{s_{k+1}^{\mathrm{up}} \,-\, s_{k+1}^{\mathrm{down}}}\; v_{k+1}^{\mathrm{down}} \\ &=: \; w_{\mathrm{up}}\, v_{k+1}^{\mathrm{up}} \;+\; w_{\mathrm{down}}\, v_{k+1}^{\mathrm{down}} \end{aligned}$$

with weights w_{up} and w_{down} which apparently add up to 1 and, using the abbreviation $R := 1+r$,

$$w_{\mathrm{up}} \;=\; \frac{s_k \;-\; s_{k+1}^{\mathrm{down}}}{s_{k+1}^{\mathrm{up}} \;-\; s_{k+1}^{\mathrm{down}}} \;=\; \frac{R^{-k}S_k \;-\; R^{-(k+1)}S_{k+1}^{\mathrm{down}}}{R^{-(k+1)}(S_{k+1}^{\mathrm{up}} \;-\; S_{k+1}^{\mathrm{down}})} \;=\; \frac{RS_k \;-\; S_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} \;-\; S_{k+1}^{\mathrm{down}}}$$

or

$$w_{\text{up}} = \frac{(1+r)S_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} = \frac{(1+r)S_k - S_k(1+\text{ret}_{\text{down}})}{S_k(1+\text{ret}_{\text{up}}) - S_k(1+\text{ret}_{\text{down}})}$$

$$= \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}$$

and the theorem is proven. ■

Beispiel: Standard-Kauf-Option in einem 3-Perioden Binomialmodell $\rightarrow$ Ü-Blatt 4