week11: Kapitel 7: Die Black-Scholes Formeln, Teil2

Wir benutzen jetzt die grundlegende Pricing-Formel aus dem Theorem 7.1 vom letzten Mal,

$$V_0^{\mathrm{BS}} \;=\; e^{-rT}\int_{\mathbb{R}} H\Big(\,S_0\,e^{\sigma\sqrt{T}\,x\,+\,(r-\frac{\sigma^2}{2})T}\,\Big)\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \tag{1}$$

um die Black-Scholes Formeln herzuleiten. Dann definieren und berechnen wir noch ein paar ‘Greeks’, das sind einfach Ableitungen des Optionspreises nach den verschiedenen Parametern wie S_t, σ, r oder t. Bevor wir das machen, schreiben wir eben noch eine leichte Verallgemeinerung von (1) hin, nämlich den Black-Scholes Preis einer pfadunabhängigen Option $H = H(S_T)$ nicht zur Zeit 0, sondern an einem beliebigen Zeitpunkt $t \in [0,T]$, das grosse T bezeichnet dabei wieder die Laufzeit der Option H. Mit denselben Überlegungen wie im week10.pdf bekommt man die folgende Formel:

Folgerung 7.1: Es sei $H = H(S_T)$ die Auszahlungsfunktion einer beliebigen, pfadunabhängigen Option. Dann ist der Zeit-t Black-Scholes Preis dieser Option gegeben durch das eindimensionale Integral

$$V_t^{\mathrm{BS}} \;=\; e^{-r(T-t)}\int_{\mathbb{R}} H\Big(\,S_t\,e^{\sigma\sqrt{T-t}\,x\,+\,(r-\frac{\sigma^2}{2})(T-t)}\,\Big)\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}}\;. \tag{2}$$

Der Optionspreis $V_t = V_t^{\mathrm{BS}}$, im Folgenden wollen wir den Superscript ‘BS’ weglassen, ist also eine Funktion von den folgenden Parametern:

$$V_t \;=\; V(S_t,\sigma,r,t,T) \tag{3}$$

Was die Abhängigkeit von den Zeit-Parametern angeht, könnte man auch schreiben $V = V(T-t)$, aber das θ in der Definition 7.1 gleich weiter unten ist die partielle Ableitung von (3) nach t, nicht die Ableitung von V nach der Restlaufzeit $\tau = T-t$, das unterscheidet sich dann um ein Vorzeichen. Die Options-Sensitivitäten oder kurz ‘Greeks’, (weil diese Grössen typischerweise mit griechischen Buchstaben bezeichnet werden) sind dann wie folgt definiert:

Definition 7.1: Let $H = H(S_T)$ be some option payoff and let

$$V_t \;=\; V(S_t,\sigma,r,t,T) \tag{4}$$

be the option price at time t given by formula (2) above. Here S_t is the asset price at time t, σ denotes the volatility and r denotes the interest rates. Then the following abbreviations are standard in the derivatives community (listed with decreasing importance):

$$\delta := \tfrac{\partial V_t}{\partial S_t} \quad \text{(delta)} \tag{5}$$

$$vega := \tfrac{\partial V_t}{\partial \sigma} \quad \text{(vega)} \tag{6}$$

$$\rho := \tfrac{\partial V_t}{\partial r} \quad \text{(rho)} \tag{7}$$

$$\theta := \tfrac{\partial V_t}{\partial t} \quad \text{(theta)} \tag{8}$$

$$\gamma := \tfrac{\partial^2 V_t}{\partial S_t^2} \quad \text{(gamma)} \tag{9}$$

Theorem 7.2 (Black-Scholes Formeln): Consider a standard call and put option with strike K, maturity T and payoffs

$$H_{\text{call}} = \max\{\, S_T - K \,,\, 0\,\} \tag{10}$$

$$H_{\text{put}} = \max\{\, K - S_T \,,\, 0\,\} \tag{11}$$

and let $V_{\text{call},t}$ and $V_{\text{put},t}$ denote their time t theoretical fair values in the Black-Scholes model. Let

$$N(x) := \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \tfrac{dy}{\sqrt{2\pi}} \tag{12}$$

denote the cumulative normal distribution function and define the quantities

$$d_\pm := \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{13}$$

such that $d_+ - d_- = \sigma\sqrt{T-t}$. Then there are the following formulae:

Option Price:

$$V_{\text{call},t} = S_t N(d_+) - K e^{-r(T-t)} N(d_-) \tag{14}$$

$$V_{\text{put},t} = -S_t N(-d_+) + K e^{-r(T-t)} N(-d_-) \tag{15}$$

Delta:

$$\delta_{\text{call}} = N(d_+) \tag{16}$$

$$\delta_{\text{put}} = -N(-d_+) \tag{17}$$

Vega:

$$vega_{\text{call}} = S_t N'(d_+)\sqrt{T-t} \tag{18}$$

$$vega_{\text{put}} = vega_{\text{call}} \tag{19}$$

Rho:

$$\rho_{\text{call}} = K\,(T-t)\,e^{-r(T-t)}\,N(d_-) \tag{20}$$
$$\rho_{\text{put}} = -K\,(T-t)\,e^{-r(T-t)}\,N(-d_-) \tag{21}$$

Theta:

$$\theta_{\text{call}} = -S_t\,\frac{\sigma}{2\sqrt{T-t}}\,N'(d_+) \;-\; r\,K\,e^{-r(T-t)}\,N(d_-) \tag{22}$$
$$\theta_{\text{put}} = -S_t\,\frac{\sigma}{2\sqrt{T-t}}\,N'(-d_+) \;+\; r\,K\,e^{-r(T-t)}\,N(-d_-) \tag{23}$$

Gamma:

$$\gamma_{\text{call}} = \frac{N'(d_+)}{\sigma\sqrt{T-t}S_t} \tag{24}$$
$$\gamma_{\text{put}} = \gamma_{\text{call}} \tag{25}$$

Proof: In the Black-Scholes model, the time t theoretical fair value of some european option with maturity $T \geq t$ and payoff $H = H(S_T)$ is given by

$$V_t = e^{-r(T-t)}\int_{\mathbb{R}} H\big(\,S_t\,e^{\sigma\sqrt{T-t}\,x\,+\,(r-\frac{\sigma^2}{2})(T-t)}\,\big)\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \tag{26}$$

Let us abbreviate

$$\tau := T-t \tag{27}$$
$$a := \sigma\sqrt{\tau} \tag{28}$$
$$b := (r-\sigma^2/2)\tau \tag{29}$$

and define x_0 as the solution of the equation

$$S_t\,e^{\sigma\sqrt{T-t}\,x\,+\,(r-\frac{\sigma^2}{2})(T-t)} = S_t\,e^{a\,x\,+\,b} \stackrel{!}{=} K \tag{30}$$

That is,

$$x_0 = \tfrac{1}{a}\log(K/S_t) - \tfrac{b}{a} \tag{31}$$

Then

$$\begin{aligned}
V_{\text{call},t} &= e^{-r\tau}\int_{\mathbb{R}} H_{\text{call}}\big(\,S_t\,e^{a\,x\,+\,b}\,\big)\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau}\int_{\mathbb{R}} \max\big\{\,S_t e^{a\,x+b} - K\,,\,0\,\big\}\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau}\int_{x_0}^{\infty}\big(\,S_t\,e^{a\,x\,+\,b} \;-\; K\,\big)\,e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau}\,S_t\,e^{b+\frac{a^2}{2}}\int_{x_0}^{\infty} e^{-\frac{1}{2}(x-a)^2}\,\frac{dx}{\sqrt{2\pi}} \;-\; e^{-r\tau}K\int_{x_0}^{\infty} e^{-\frac{x^2}{2}}\,\frac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau}\;S_t\;e^{b+\frac{a^2}{2}}\;N(-x_0+a) \;-\; e^{-r\tau}KN(-x_0) \\
&= S_t\,N(d_+) \;-\; e^{-r(T-t)}KN(d_-)
\end{aligned} \tag{32}$$

since

$$-x_0 \;=\; \tfrac{1}{a}\log(S_t/K) + \tfrac{b}{a} \;=\; \tfrac{\log(S_t/K)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \;=\; d_-$$

and $-x_0 + a = d_- + \sigma\sqrt{\tau} = d_+$. Furthermore

$$-r\tau + b + a^2/2 \;=\; -r\tau + (r-\sigma^2/2)\tau + (\sigma\sqrt{\tau})^2/2 \;=\; 0$$

which results in the last line of (32). The proof for the put fair value is similar. In particular, since

$$N(d) + N(-d) \;=\; 1$$

we get from (14,15)

$$V_{\text{call},t} - V_{\text{put},t} \;=\; S_t - e^{-r(T-t)}K \tag{33}$$

This relation is called 'call-put parity'.

Delta: To compute the delta we use the third line of (32) instead of the last one. We get

$$\begin{aligned}
\delta_{\text{call}}(S_t,t) &= \frac{\partial}{\partial S_t}\left\{ e^{-r\tau}\int_{x_0}^{\infty}(S_t\, e^{ax+b}-K)\, e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}\right\} \\
&= e^{-r\tau}\int_{x_0}^{\infty} e^{ax+b}\, e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \;-\; e^{-r\tau}\underbrace{(S_t\, e^{ax_0+b}-K)}_{=0}\,\frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}}\frac{\partial x_0}{\partial S_t} \\
&= N(d_+)
\end{aligned} \tag{34}$$

since the above integral is the same as the first integral in (32), up to the factor of S_t which has been differentiated away. Finally it follows from call-put parity that

$$\delta_{\text{put}}(S,t) \;=\; N(d_+) - 1 \;=\; N(-d_+)\;.$$

Vega: Again we use the third line of (32):

$$\begin{aligned}
vega_{\text{call}}(S_t,t) &= \frac{\partial}{\partial\sigma}\left\{ e^{-r\tau}\int_{x_0}^{\infty}(S_t\, e^{ax+b}-K)\, e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}\right\} \\
&= e^{-r\tau}\int_{x_0}^{\infty} S_t\big(\tfrac{\partial a}{\partial\sigma}x + \tfrac{\partial b}{\partial\sigma}\big)e^{ax+b}\, e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \;-\; e^{-r\tau}\underbrace{(S_t\, e^{ax_0+b}-K)}_{=0}\,\frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}}\frac{\partial x_0}{\partial\sigma} \\
&= e^{-r\tau}e^{b+\frac{a^2}{2}}S_t\int_{x_0}^{\infty}(\sqrt{\tau}x-\sigma\tau)\, e^{-\frac{(x-a)^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\int_{x_0}^{\infty}(x-a)\, e^{-\frac{(x-a)^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\left[-\frac{e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}}\right]_{x_0}^{\infty} \;=\; S_t\sqrt{\tau}\,\frac{e^{-\frac{(x_0-a)^2}{2}}}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\,\frac{e^{-\frac{(-d_+)^2}{2}}}{\sqrt{2\pi}} \;=\; S_t\sqrt{\tau}\,\frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} \;=\; S_t\sqrt{\tau}\,N'(d_+)
\end{aligned} \tag{35}$$

The vega for the put follows again from call-put parity. The remaining formulae may be looked up, for example, in the book on Quantitative Finance, Vol.1, by Paul Wilmott. ■