Hochschule RheinMain
Prof. Dr. D. Lehmann WS 2023/24

Lösungen zum 9. Übungsblatt
Einführung in die Finanzmathematik

Aufgabe 1: a) Wir haben

$$N - k = N - \frac{N + \sqrt{N}\,x}{2} = \frac{N - \sqrt{N}\,x}{2}$$

und

$$\begin{aligned}
\frac{1}{2^N}\binom{N}{k} &= \frac{1}{2^N}\,\frac{N!}{k!\,(N-k)!} \\
&\overset{\text{Stirling}}{\approx} \frac{1}{2^N}\,\frac{\sqrt{2\pi N}\,\left(\frac{N}{e}\right)^N}{\sqrt{2\pi k}\,\left(\frac{k}{e}\right)^k\sqrt{2\pi(N-k)}\,\left(\frac{N-k}{e}\right)^{N-k}} \\
&= \frac{1}{2^N}\,\frac{1}{\sqrt{2\pi}}\sqrt{\frac{N}{k(N-k)}}\,\frac{N^N}{k^k(N-k)^{N-k}}
\end{aligned}$$

b)

$$\frac{N}{k} = \frac{N}{\frac{N+\sqrt{N}\,x}{2}} = \frac{2}{1+\frac{x}{\sqrt{N}}}\,,$$

$$\frac{N}{N-k} = \frac{N}{\frac{N-\sqrt{N}\,x}{2}} = \frac{2}{1-\frac{x}{\sqrt{N}}}$$

c)

$$\begin{aligned}
\frac{1}{2^N}\,\frac{N^N}{k^k(N-k)^{N-k}} &= \frac{1}{2^N}\left(\frac{N}{k}\right)^k\left(\frac{N}{N-k}\right)^{N-k} \\
&= \frac{1}{2^N}\left(\frac{2}{1+\frac{x}{\sqrt{N}}}\right)^k\left(\frac{2}{1-\frac{x}{\sqrt{N}}}\right)^{N-k} \\
&= \left(\frac{1}{1+\frac{x}{\sqrt{N}}}\right)^k\left(\frac{1}{1-\frac{x}{\sqrt{N}}}\right)^{N-k} \\
&= \left(\frac{1}{1+\frac{x}{\sqrt{N}}}\right)^{\frac{N+\sqrt{N}\,x}{2}}\left(\frac{1}{1-\frac{x}{\sqrt{N}}}\right)^{\frac{N-\sqrt{N}\,x}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \quad \frac{1}{(1-\frac{x^2}{N})^{N/2}} \times \frac{(1-\frac{x}{\sqrt{N}})^{\frac{\sqrt{N}\,x}{2}}}{(1+\frac{x}{\sqrt{N}})^{\frac{\sqrt{N}\,x}{2}}} \\
&\stackrel{N\to\infty}{\to} \quad \frac{1}{e^{-\frac{x^2}{2}}} \times \frac{(e^{-x})^{x/2}}{(e^{x})^{x/2}} \;=\; e^{-\frac{x^2}{2}}
\end{aligned}
$$

d) Schliesslich, mit

$$
\Delta x \;:=\; x_{N,k} - x_{N,k-1} \;=\; \frac{2k-N}{\sqrt{N}} - \frac{2(k-1)-N}{\sqrt{N}} \;=\; \frac{2}{\sqrt{N}}
$$

bekommen wir

$$
\begin{aligned}
\frac{\sqrt{\frac{N}{k(N-k)}}}{\Delta x} &= \frac{\sqrt{\frac{N}{(\frac{N+\sqrt{N}\,x}{2})(\frac{N-\sqrt{N}\,x}{2})}}}{\frac{2}{\sqrt{N}}} \\
&= \frac{\sqrt{\frac{4N}{N^2-N\,x^2}}}{\frac{2}{\sqrt{N}}} \\
&= \sqrt{\frac{N^2}{N^2-N\,x^2}} \\
&= \sqrt{\frac{1}{1-\frac{x^2}{N}}} \quad \stackrel{N\to\infty}{\to} \quad 1
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{1}{2^N}\binom{N}{k} &\approx \frac{1}{\sqrt{2\pi}} \times \frac{1}{2^N}\,\frac{N^N}{k^k(N-k)^{N-k}} \times \sqrt{\frac{N}{k(N-k)}} \\
&\approx \frac{1}{\sqrt{2\pi}} \times e^{-\frac{x^2}{2}} \times \Delta x\,.
\end{aligned}
$$

Aufgabe 2: **a)** Es ist

$$
\begin{aligned}
\int_0^\infty x^N e^{-x} dx &= x^N e^{-x}\big|_0^\infty - \int_0^\infty N x^{N-1}(-e^{-x}) dx \\
&= N\int_0^\infty x^{N-1} e^{-x} dx \\
&= N(N-1)\int_0^\infty x^{N-2} e^{-x} dx \\
&\;\vdots \\
&= N(N-1)\cdots 2\cdot 1 \int_0^\infty x^0 e^{-x} dx \;=\; N!
\end{aligned}
$$

b) Mit der Substitution $x = y + N$ bekommen wir

$$\begin{aligned} N! &= \int_0^\infty x^N e^{-x} dx \\ &= \int_{-N}^\infty (y+N)^N e^{-y-N} dy \\ &= \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty \left(\tfrac{y+N}{N}\right)^N e^{-y} dy \\ &= \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty e^{N\log[1+\frac{y}{N}]\,-\,y}\, dy \end{aligned}$$

c) Und mit der Taylor-Entwicklung $\log(1+h) \approx h - \frac{h^2}{2}$ erhalten wir

$$\begin{aligned} N! &= \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty e^{N\log[1+\frac{y}{N}]\,-\,y}\, dy \\ &\approx \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty e^{N\left[\frac{y}{N}-\frac{1}{2}(\frac{y}{N})^2\right]\,-\,y}\, dy \\ &= \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty e^{-\frac{y^2}{2N}} dy \end{aligned}$$

d)

$$\begin{aligned} N! &\approx \left(\tfrac{N}{e}\right)^N \int_{-N}^\infty e^{-\frac{y^2}{2N}} dy \\ &= \left(\tfrac{N}{e}\right)^N \int_{-\sqrt{N}}^\infty e^{-\frac{z^2}{2}} dz\,\sqrt{N} \\ &= \sqrt{2\pi N}\left(\tfrac{N}{e}\right)^N \times \int_{-\sqrt{N}}^\infty e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\ &\approx \sqrt{2\pi N}\left(\tfrac{N}{e}\right)^N \times \int_{-\infty}^\infty e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\ &= \sqrt{2\pi N}\left(\tfrac{N}{e}\right)^N \times\ 1\ . \end{aligned}$$