

Lösungen zum 7. Übungsblatt
Einführung in die Finanzmathematik

1. Aufgabe: Wir benutzen die Formel aus dem Theorem 5.2 mit Zinsen $r = 0$,

$$\begin{aligned} V_0 &= (1+0)^{-N} \times \sum_{\ell=0}^N H(S_{N,\ell}) \times \binom{N}{\ell} p_{rn}^\ell (1-p_{rn})^{N-\ell} \\ &= \sum_{\ell=0}^N H(S_{N,\ell}) \times \binom{N}{\ell} p_{rn}^\ell (1-p_{rn})^{N-\ell} \end{aligned}$$

mit der risikoneutralen W'keit

$$p_{rn} = \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \stackrel{r=0}{=} \frac{-\text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} = \frac{-(-q)}{q - (-q)} = \frac{1}{2}$$

und den Underlyingpreisen

$$\begin{aligned} S_{N,\ell} &= S_0 (1 + \text{ret}_{\text{up}})^\ell (1 + \text{ret}_{\text{down}})^{N-\ell} \\ &= S_0 (1 + q)^\ell (1 - q)^{N-\ell} \end{aligned}$$

Also,

$$H(S_{N,\ell}) = S_{N,\ell}^2 = S_0^2 \times (1+q)^{2\ell} (1-q)^{2(N-\ell)}$$

und wir bekommen

$$\begin{aligned} V_0 &= \sum_{\ell=0}^N H(S_{N,\ell}) \times \binom{N}{\ell} \left(\frac{1}{2}\right)^\ell \left(1 - \frac{1}{2}\right)^{N-\ell} \\ &= S_0^2 \times \frac{1}{2^N} \sum_{\ell=0}^N \binom{N}{\ell} [(1+q)^2]^\ell [(1-q)^2]^{N-\ell} \\ &= S_0^2 \times \frac{1}{2^N} \times \left\{ (1+q)^2 + (1-q)^2 \right\}^N \\ &= S_0^2 \times \frac{1}{2^N} \times \left\{ 1 + 2q + q^2 + 1 - 2q + q^2 \right\}^N \\ &= S_0^2 \times \frac{1}{2^N} \times \left\{ 2 + 2q^2 \right\}^N \\ &= S_0^2 \times \left\{ 1 + q^2 \right\}^N . \end{aligned}$$

2. Aufgabe: Nach Theorem 5.2 gilt mit Zinsen $r = 0$

$$V_0 = \sum_{\ell=0}^N H(S_{N,\ell}) \times \binom{N}{\ell} p_{\text{rn}}^{\ell} (1 - p_{\text{rn}})^{N-\ell}$$

mit den Underlyingpreisen

$$S_{N,\ell} = S_0 (1 + \text{ret}_{\text{up}})^{\ell} (1 + \text{ret}_{\text{down}})^{N-\ell}$$

Wir haben dann

$$H(S_{N,\ell}) = \frac{S_0}{S_{N,\ell}} = \frac{1}{(1 + \text{ret}_{\text{up}})^{\ell} (1 + \text{ret}_{\text{down}})^{N-\ell}}$$

und bekommen damit

$$\begin{aligned} V_0 &= \sum_{\ell=0}^N \frac{1}{(1 + \text{ret}_{\text{up}})^{\ell} (1 + \text{ret}_{\text{down}})^{N-\ell}} \times \binom{N}{\ell} p_{\text{rn}}^{\ell} (1 - p_{\text{rn}})^{N-\ell} \\ &= \sum_{\ell=0}^N \binom{N}{\ell} \left(\frac{p_{\text{rn}}}{1 + \text{ret}_{\text{up}}} \right)^{\ell} \left(\frac{1 - p_{\text{rn}}}{1 + \text{ret}_{\text{down}}} \right)^{N-\ell} \\ &= \left\{ \frac{p_{\text{rn}}}{1 + \text{ret}_{\text{up}}} + \frac{1 - p_{\text{rn}}}{1 + \text{ret}_{\text{down}}} \right\}^N \end{aligned}$$

Dabei haben wir in der letzten Zeile die allgemeine Binomische Formel

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

benutzt.