Lösungen zum 6. Übungsblatt
Einführung in die Finanzmathematik

1. Aufgabe: a) From the definition of the returns

$$\mathrm{ret}_k \;:=\; \frac{S_k - S_{k-1}}{S_{k-1}}$$

we get

$$S_N \;=\; S_{N-1}(1+\mathrm{ret}_N)$$

and by induction

$$S_N \;=\; S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)$$

Since all the returns are independent,

$$\begin{aligned} \mathsf{E}[\,S_N\,] &= \mathsf{E}\Big[S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)\Big] \\ &= S_0 \prod_{k=1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_k]\big)\;. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathsf{E}[\mathrm{ret}_k] &= +q \times p_{\mathrm{up}} \;+\; (-q)\times p_{\mathrm{down}} \\ &= q/2 - q/2 \;=\; 0 \end{aligned}$$

Thus,

$$\mathsf{E}[\,S_N\,] \;=\; S_0 \prod_{k=1}^{N} \big(1+0\big) \;=\; S_0\;.$$

b) The notation $\mathsf{E}\big[\cdots \,\big|\, \{S_j\}_{j=0}^{k}\,\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned} S_N &= S_0 \prod_{j=1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_0 \prod_{j=1}^{k} \big(1+\mathrm{ret}_j\big) \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \\ &= S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \end{aligned}$$

and obtain

$$\begin{aligned}
\mathsf{E}\big[\, S_N \,\big|\, \{S_j\}_{j=0}^k \,\big] &= \mathsf{E}\Big[\, S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \,\Big|\, \{S_j\}_{j=0}^k \,\Big] \\
&= S_k \prod_{j=k+1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_j]\big) \\
&= S_k \prod_{j=k+1}^{N} \big(1+0\big) \;=\; S_k \;.
\end{aligned}$$

c) Because of part (a), we have

$$\begin{aligned}
\mathsf{E}\Big[\, \frac{1}{N} \sum_{m=1}^{N} S_m \,\Big] &= \frac{1}{N} \sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big] \\
&= \frac{1}{N} \sum_{m=1}^{N} S_0 \;=\; S_0 \;.
\end{aligned}$$

d) Because of part (b), we obtain

$$\begin{aligned}
\mathsf{E}\Big[\, \frac{1}{N} \sum_{m=1}^{N} S_m \,\Big|\, \{S_j\}_{j=0}^k \,\Big] &= \frac{1}{N} \sum_{m=1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k \,\big] \\
&= \frac{1}{N} \Big\{ \sum_{m=1}^{k} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k \,\big] \;+\; \sum_{m=k+1}^{N} \mathsf{E}\big[\, S_m \,\big|\, \{S_j\}_{j=0}^k \,\big] \Big\} \\
&= \frac{1}{N} \Big\{ \sum_{m=1}^{k} S_m \;+\; \sum_{m=k+1}^{N} S_k \Big\} \\
&= \frac{k}{N} \times \frac{1}{k} \sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N} \times S_k \;.
\end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N \;=\; S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} \;=\; \prod_{k=1}^{N} \frac{1}{1+\mathrm{ret}_k} \;.$$

Since all the returns are independent,

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \,\big] &= \mathsf{E}\Big[\prod_{k=1}^{N} \frac{1}{1+\mathrm{ret}_k} \Big] \\
&= \prod_{k=1}^{N} \mathsf{E}\Big[\frac{1}{1+\mathrm{ret}_k} \Big] \\
&= \prod_{k=1}^{N} \Big\{ \frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2} \Big\} \\
&= \prod_{k=1}^{N} \Big\{ \frac{1}{1-q^2} \Big\} \\
&= \frac{1}{(1-q^2)^N} \;.
\end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots \big| \{S_j\}_{j=0}^k\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write as in part (b)

$$S_0 \;/\; S_N \;=\; S_0 \;/\; \Big\{S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big)\Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \,\big|\, \{S_j\}_{j=0}^k\big] &= S_0/S_k \; \mathsf{E}\left[\prod_{m=k+1}^{N} \frac{1}{1+\mathrm{ret}_m} \,\Big|\, \{S_j\}_{j=0}^k\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_m}\right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2}\right\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{\frac{1}{1-q^2}\right\} \\
&= S_0/S_k \; \frac{1}{(1-q^2)^{N-k}} \,. \qquad \blacksquare
\end{aligned}$$