Lösungen zum 11. Übungsblatt
Einführung in die Finanzmathematik

Aufgabe 1: Using the Black-Scholes formulae, we get

$$\begin{aligned} d_\pm &= \frac{\log[S_0/K] \,+\, (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\log[100/120] \,+\, (0.03 \pm (0.25)^2/2)4}{0.25 \times \sqrt{4}} \end{aligned}$$

which gives

$$\begin{aligned} d_+ &= 0.125357 \\ d_- &= -0.37464 \end{aligned}$$

In the exam, you will get a table with $N(x)$-values from which you can read off the numbers (probably with less digits, then giving a more unprecise result)

$$\begin{aligned} N(d_+) &= 0.549879 \\ N(d_-) &= 0.353963 \end{aligned}$$

Thus,

$$\begin{aligned} V_0 &= S_0 N(d_+) - K e^{-rT} N(d_-) \\ &= 100 \times 0.549879 - 120\, e^{-0.12} \times 0.353963 \;=\; 17.3155\,. \end{aligned}$$

Aufgabe 2: Nach den Black-Scholes Formeln ist

$$\begin{aligned} V^{\mathrm{BS}}_{\mathrm{Call},0} &= S_0 N(d_+) \,-\, K e^{-rT} N(d_-) \\ V^{\mathrm{BS}}_{\mathrm{Put},0} &= -S_0 N(-d_+) \,+\, K e^{-rT} N(-d_-) \end{aligned}$$

so dass

$$\begin{aligned} V^{\mathrm{BS}}_{\mathrm{Call},0} \,-\, V^{\mathrm{BS}}_{\mathrm{Put},0} &= S_0 \underbrace{\big[N(d_+) + N(-d_+)\big]}_{=1} - K e^{-rT} \underbrace{\big[N(d_-) + N(-d_-)\big]}_{=1} \\ &= S_0 \,-\, K e^{-rT} \end{aligned}$$

da wegen

$$\begin{aligned} N(-x) &= \int_{-\infty}^{-x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{-x} e^{-\frac{(-y)^2}{2}} (-1) \frac{d(-y)}{\sqrt{2\pi}} \\ &\overset{v:=-y}{=} \int_{+\infty}^{+x} e^{-\frac{v^2}{2}} (-1) \frac{dv}{\sqrt{2\pi}} \\ &= \int_{x}^{\infty} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \end{aligned}$$

die Gleichung

$$\begin{aligned} N(x) + N(-x) &= \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} + \int_{-\infty}^{-x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} + \int_{x}^{\infty} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = 1 \end{aligned}$$

erfüllt ist.

Aufgabe 3: **a)** We have

$$\begin{aligned} H_{\text{call,perf}}(S_T) &= \max\{\, S_T/S_0 - 1\,,\, 0\,\} \\ &= \tfrac{1}{S_0} \max\{\, S_T - S_0\,,\, 0\,\} \\ &=: \tfrac{1}{S_0} H_{\text{call,abs}}(S_T) \end{aligned}$$

where $H_{\text{call,abs}}(S_T) = H_{\text{call}}(S_T)$ is just a standard call in absolute amount to which the Black-Scholes formula can be applied. Thus, since $K = S_0$ and $r = 0$,

$$\begin{aligned} V_0 &= \text{price}\big(H_{\text{call,perf}}\big) \\ &= \tfrac{1}{S_0} \text{price}\big(H_{\text{call,abs}}\big) \\ &= \tfrac{1}{S_0} \big\{ S_0\, N(d_+) - S_0\, N(d_-) \big\} \\ &= N(d_+) - N(d_-) \end{aligned}$$

with

$$\begin{aligned} d_\pm &= \frac{\log\left[\frac{S_0}{S_0}\right] + (0 \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \pm \frac{\sigma\sqrt{T}}{2}\,. \end{aligned}$$

This proves part (a).

b) Using (a), we can write

$$
\begin{aligned}
V_0 &= N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2) \\
&= \frac{N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2)}{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)} \times \left\{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)\right\} \\
&\approx N'(0) \times \sigma\sqrt{T} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} \times \sigma\sqrt{T} \\
&= \frac{1}{\sqrt{2\pi}} \times \sigma\sqrt{T} \\
&= 0.398942.. \times \sigma\sqrt{T} \\
&\approx 0.4 \times \sigma\sqrt{T}\ .
\end{aligned}
$$