week8b: Erwartungswerte mit Brownschen Bewegungen

Wir wollen uns jetzt überlegen, wie man allgemeine Erwartungswerte mit Brownschen Bewegungen konkret berechnen kann. Das Setting war das folgende: Consider some discrete times t_k in the intervall $[0,T]$,

$$t_k \;=\; k\tfrac{T}{N} \;=\; k\Delta t\,, \qquad k=0,1,...,N \tag{1}$$

where

$$N \;=\; N_T \;:=\; \tfrac{T}{\Delta t} \;\in\; \mathbb{N} \tag{2}$$

and let

$$\phi_1\;,\;\phi_2\;,\;\cdots\;,\;\phi_N \tag{3}$$

be independent normally distributed random numbers with mean zero and variance one,

$$\phi_k \in \mathcal{N}(0,1) \quad \text{i.i.d.} \tag{4}$$

Then a Brownian motion at discretized time $t = t_k = k\Delta t$ is given by the combination of integration variables

$$x_t \;=\; x_{t_k} \;:=\; \sqrt{\Delta t}\,\sum_{j=1}^{k}\phi_j \tag{5}$$

We start with the following expectation which involves only one Brownian motion observed at one single time $t = T = t_N$:

$$\begin{aligned}\mathsf{E}\big[\,f(x_T)\,\big] \;=\; \mathsf{E}\big[\,f(x_{t_N})\,\big] \;&=\; \mathsf{E}\Big[f\Big(\sqrt{\Delta t}\,\textstyle\sum_{k=1}^N\phi_k\Big)\Big]\\ &=\; \int_{\mathbb{R}^N} f\Big(\sqrt{\Delta t}\,\textstyle\sum_{k=1}^N\phi_k\Big)\prod_{k=1}^N \tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{\phi_k^2}{2}}\,d\phi_k\end{aligned} \tag{6}$$

Here f is an arbitrary function. We make a substitution of variables $(\phi_k)_{1\le k\le N} \to (x_k)_{1\le k\le N}$ defined as follows:

$$\begin{array}{lcl} x_1 &=& \sqrt{\Delta t}\,\phi_1 \\ x_2 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2) \\ x_3 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+\phi_3) \\ &\vdots& \\ x_N &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+...+\phi_N) \end{array} \quad\Leftrightarrow\quad \begin{array}{lcl} \phi_1 &=& x_1/\sqrt{\Delta t} \\ \phi_2 &=& (x_2-x_1)/\sqrt{\Delta t} \\ \phi_3 &=& (x_3-x_2)/\sqrt{\Delta t} \\ &\vdots& \\ \phi_N &=& (x_N-x_{N-1})/\sqrt{\Delta t} \end{array} \tag{7}$$

The Jacobian of the transformation (7) is $\det \frac{\partial\phi}{\partial x} = 1/\sqrt{\Delta t}^N$ since

$$\frac{\partial\phi}{\partial x} = \begin{pmatrix} - & \nabla_x\,\phi_1 & - \\ & \vdots & \\ - & \nabla_x\,\phi_N & - \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_N} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\phi_N}{\partial x_1} & \frac{\partial\phi_N}{\partial x_2} & \cdots & \frac{\partial\phi_N}{\partial x_N} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} \end{pmatrix}$$

Thus the expectation (6) becomes, with $N = N_t = t/\Delta t$,

$$\begin{aligned} \mathsf{E}\Big[f\Big(\sqrt{\Delta t}\,\textstyle\sum_{k=1}^N \phi_k\Big)\Big] &= \int_{\mathbb{R}^N} f\Big(\sqrt{\Delta t}\,\textstyle\sum_{k=1}^N \phi_k\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}}\, d\phi_k \Big\} \\ &= \int_{\mathbb{R}^N} f\Big(\sqrt{\Delta t}\,\textstyle\sum_{k=1}^N \phi_k\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}} \Big\}\, d\phi_1 \cdots d\phi_N \\ &= \int_{\mathbb{R}^N} f(\,x_N\,) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \Big\}\, \det\Big[\,\frac{\partial\phi}{\partial x}\,\Big]\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(\,x_N\,) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \Big\}\, \frac{1}{(\sqrt{\Delta t})^N}\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(\,x_N\,) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(x_{k-1} - x_k)^2}{2\Delta t}} \Big\}\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(\,x_N\,) \prod_{k=1}^N \Big\{ p_{\Delta t}(x_{k-1}, x_k)\, dx_k \Big\} \end{aligned} \tag{8}$$

where we introduced the kernel

$$p_\tau(x,y) := \frac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \tag{9}$$

and used the definition

$$x_0 := 0 \tag{10}$$

The kernel (9) has the following basic property:

Lemma 4.1: *Let $p_t(x,y)$ be given by (9). Then*

$$\int_{\mathbb{R}} p_s(x,y)\, p_t(y,z)\, dy = p_{s+t}(x,z) \tag{11}$$

Proof: We have

$$\begin{aligned}
p_s(x,y)p_t(y,z) &= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-(\frac{1}{2s}+\frac{1}{2t})y^2+(\frac{x}{s}+\frac{z}{t})y} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y^2-2\frac{xt+zs}{s+t}y)} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2}\; e^{\frac{s+t}{2st}(\frac{xt+zs}{s+t})^2} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2}\; e^{\frac{(xt+zs)^2}{2st(s+t)}} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-x^2(\frac{1}{2s}-\frac{t}{2s(s+t)})-z^2(\frac{1}{2t}-\frac{s}{2t(s+t)})+\frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{x^2}{2(s+t)}-\frac{z^2}{2(s+t)}+\frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2}
\end{aligned}$$

Also

$$\begin{aligned}
\int_{\mathbb{R}} p_s(x,y)p_t(y,z)\,dy &= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \int_{\mathbb{R}} e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2}dy \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \int_{\mathbb{R}} e^{-\frac{s+t}{2st}v^2}dv \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \sqrt{2\pi}\sqrt{\frac{st}{s+t}} \\
&= \frac{1}{\sqrt{2\pi(s+t)}}\; e^{-\frac{(x-z)^2}{2(s+t)}} \\
&= p_{s+t}(x,z)
\end{aligned}$$

which proves the lemma. ∎

Using this lemma, we can actually perform the integrals over $x_1, x_2, \cdots, x_{N-1}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \cdots \int_{\mathbb{R}} dx_{N-1}\; \underbrace{p_{\Delta t}(x_0,x_1)\, p_{\Delta t}(x_1,x_2)}_{\int dx_1\, \to\, p_{2\Delta t}(x_0,x_2)}\, p_{\Delta t}(x_2,x_3)\; \cdots\; p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1}\; \underbrace{p_{2\Delta t}(x_0,x_2)\, p_{\Delta t}(x_2,x_3)}_{\int dx_2\, \to\, p_{3\Delta t}(x_0,x_3)}\; \cdots\; p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1}\; p_{3\Delta t}(x_0,x_3)\; \cdots\; p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_{N-1}\; p_{(N-1)\Delta t}(x_0,x_{N-1})\, p_{\Delta t}(x_{N-1},x_N) \\
&= p_{N\Delta t}(x_0,x_N)
\end{aligned}$$

Thus (8) simplifies to

$$\mathsf{E}\Big[f\Big(\sqrt{\Delta t}\sum_{k=1}^{N_t}\phi_k\Big)\Big] = \int_{\mathbb{R}^N} f(\,x_N\,)\prod_{k=1}^{N}\Big\{\,p_{\Delta t}(x_{k-1},x_k)\,dx_k\,\Big\}$$
$$= \int_{\mathbb{R}} f(\,x_N\,)\;\times\;p_{N\Delta t}(x_0,x_N)\,dx_N$$
$$\overset{\substack{N\Delta t=T\\ x_0=0}}{=} \int_{\mathbb{R}} f(x)\;\times\;\frac{1}{\sqrt{2\pi T}}\,e^{-\frac{x^2}{2T}}\,dx \tag{12}$$

Instead of labelling the x with $k \in \{1, 2, ..., N\}$, we label them with $t_k := k\Delta t$ which has the meaning of time. In particular, $t_N = N\Delta t = T$. So, we rename $x_k \to x_{k\Delta t} = x_{t_k}$. With that, we write down the following very important

Definition 4.1: *Let* $N_T = T/\Delta t$ *and* $p_t(x,y) := \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(x-y)^2}{2t}}$. *Then the measure*

$$dW(\{x_t\}_{0<t\le T}) \;:=\; \lim_{\Delta t\to 0}\prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\,dx_{k\Delta t} \tag{13}$$

is called the Wiener measure and the family of random variables or integration variables $\{x_t\}_{0<t\le T}$ *is called a Brownian motion. In terms of i.i.d. random variables* $\phi_k \in \mathcal{N}(0,1)$,

$$x_t \;:=\; \lim_{\Delta t\to 0}\sqrt{\Delta t}\sum_{k=1}^{t/\Delta t}\phi_k \tag{14}$$

Remark: The time discretized version of the Wiener measure, this is what we actually will usually use, is simply given by a product of Gaussian normal distributions. With $N = N_T = T/\Delta t$ and $t_k = k\Delta t$,

$$dW(\{x_{t_k}\}_{0<k\le N}) = \prod_{k=1}^{N} p_{\Delta t}(x_{t_{k-1}}, x_{t_k})\,dx_{t_k}$$
$$= \prod_{k=1}^{N}\Big\{\,e^{-\frac{(\,x_{t_k}-x_{t_{k-1}}\,)^2}{2\Delta t}}\;\frac{dx_{t_k}}{\sqrt{2\pi\Delta t}}\,\Big\}$$
$$= \prod_{k=1}^{N}\Big\{\,e^{-\frac{\phi_k^2}{2}}\;\frac{d\phi_k}{\sqrt{2\pi}}\,\Big\} \tag{15}$$

where the last line in (15) is due to the substitution of variables, which is equivalent to the definition of Brownian motion in discrete time,

$$x_{t_k} \;=\; \sqrt{\Delta t}\sum_{j=1}^{k}\phi_j \tag{16}$$

from which we get the recursion

$$x_{t_k} \;=\; x_{t_{k-1}} \;+\; \sqrt{\Delta t}\;\phi_k \tag{17}$$

The formulae (15,16,17) are very important and will be used over and over again.

Integrals with respect to the Wiener measure are computed according to the following

Theorem 4.1: *Let $F : \mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$ be some times and $x_{t_1}, ..., x_{t_m}$ be a Brownian motion observed at those times. Then*

$$\begin{aligned} \mathsf{E}[\, F(x_{t_1}, ..., x_{t_m})\,] &\overset{\text{notation}}{=} \int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) \\ &\overset{\substack{\text{statement}\\ \text{theorem}}}{=} \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \end{aligned} \tag{18}$$

Proof: The calculation is very similar to the calculation which lead to (12). Because of (11) only the $x_{t_1}, ..., x_{t_m}$ integration variables survive. We are a bit less explicit and a bit more compact now and write

$$\begin{aligned} &\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) \\ &= \lim_{\Delta t \to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \\ &= \lim_{\Delta t \to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_{t_1}} p_{\Delta t}(...)\, dx_{k\Delta t} \prod_{k=N_{t_1}+1}^{N_{t_2}} p_{\Delta t}(...)\, dx_{k\Delta t} \times \cdots \\ &\qquad \cdots \times \prod_{k=N_{t_{m-1}}+1}^{N_{t_m}} p_{\Delta t}(...)\, dx_{k\Delta t} \\ &\overset{(11)}{=} \lim_{\Delta t \to 0} \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m})\, p_{t_1}(x_0, x_{t_1})\, dx_{t_1}\; p_{t_2 - t_1}(x_{t_1}, x_{t_2})\, dx_{t_2} \;\times \cdots \\ &\qquad \cdots \times\; p_{t_m - t_{m-1}}(x_{t_{m-1}}, x_{t_m})\, dx_{t_m} \\ &= \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \end{aligned}$$

which coincides with (18). ■