week7: Kapitel 4: Brownsche Bewegung, Wiener-Maß und das Black-Scholes Modell, Teil2

Letztes Mal hatten wir für den Preisprozess die folgende Darstellung hergeleitet:

$$S_t \;=\; S_0\,\exp\Big\{\,\sigma\sqrt{\Delta t}\sum_{k=1}^{N_t}\phi_k \;-\; \frac{\sigma^2}{2}\Delta t\sum_{k=1}^{N_t}\phi_k^2 \;+\; O(\sqrt{\Delta t})\,\Big\} \tag{1}$$

The first term in the exponent converges to a Brownian motion $x_t = \lim_{\Delta t\to 0}\sqrt{\Delta t}\,\sum_{k=1}^{N_t}\phi_k$ and the last term vanishes, but what about the second term? There is the following

Theorem 4.2: Let

$$I_{\Delta t} \;:=\; \Delta t\sum_{k=1}^{N_t}\phi_k^2$$

with $N_t = t/\Delta t$ and $\phi_1, \phi_2, \cdots$ being independent Gaussian random numbers with mean 0 and standard deviation 1. Then the following statements hold:

a) $\mathsf{E}[I_{\Delta t}] \;=\; t$

b) $\mathsf{V}[I_{\Delta t}] \;=\; 2\,t\,\Delta t$

c) $\lim\limits_{\Delta t\to 0}\mathsf{Prob}\Big[\,\big|I_{\Delta t} - t\,\big| \geq \varepsilon\,\Big] \;=\; 0 \qquad \forall\varepsilon > 0\;.$

More intuitively, we may rewrite the statement of part (c) simply as

$$\lim_{\Delta t\to 0} I_{\Delta t} \;=\; \lim_{\Delta t\to 0}\Delta t\sum_{k=1}^{N_t}\phi_k^2 \;=\; t\;.$$

Proof: For standard normal distributed random numbers we have

$$\mathsf{E}[\phi^2] \;=\; 1$$

$$\mathsf{E}[\phi^4] \;=\; 3$$

$$\mathsf{V}[\phi^2] \;=\; \mathsf{E}[\phi^4] - \big(\mathsf{E}[\phi^2]\big)^2 \;=\; 2$$

since more generally

$$\mathsf{E}[\phi^{2n}] \;=\; (2n-1)!!$$

Thus we get

$$\begin{aligned}
\mathsf{E}[I_{\Delta t}] &= \Delta t \sum_{k=1}^{N_t} \mathsf{E}[\phi_k^2] \\
&= \Delta t \sum_{k=1}^{N_t} 1 \\
&= \Delta t \, N_t \; = \; t \quad .
\end{aligned}$$

To calculate the variance, we rewrite it as a covariance since the covariance is a bilinear quantity where we can bring sums from inside to the outside of the covariance as follows:

$$\begin{aligned}
\mathsf{V}[I_{\Delta t}] &= \mathsf{Cov}[I_{\Delta t}, I_{\Delta t}] \\
&= \mathsf{Cov}\Big[\; \Delta t \sum_{k=1}^{N_t} \phi_k^2 \; , \; \Delta t \sum_{\ell=1}^{N_t} \phi_\ell^2 \;\Big] \\
&= (\Delta t)^2 \sum_{k,\ell=1}^{N_t} \mathsf{Cov}[\; \phi_k^2 \; , \; \phi_\ell^2 \;] \\
&= (\Delta t)^2 \sum_{k,\ell=1}^{N_t} \delta_{k,\ell} \, \mathsf{Cov}[\; \phi_k^2 \; , \; \phi_k^2 \;] \\
&= (\Delta t)^2 \sum_{k=1}^{N_t} \mathsf{V}[\; \phi_k^2 \;] \\
&= (\Delta t)^2 \sum_{k=1}^{N_t} 2 \\
&= 2 \, t \, \Delta t \; .
\end{aligned}$$

Now we use Chebyshev's inequality. It states that for any random variable X we have

$$\mathsf{Prob}\Big(\; \big| \, X \; - \; \mathsf{E}[X] \, \big| \; \geq \; \varepsilon \; \Big) \quad \leq \quad \frac{\mathsf{V}[X]}{\varepsilon^2} \quad .$$

Then we put $X = I_{\Delta t}$ such that with the results from part (a) and (b) we obtain

$$\mathsf{Prob}\Big(\; \big| \, I_{\Delta t} \; - \; t \, \big| \; \geq \; \varepsilon \; \Big) \quad \leq \quad \frac{2t\Delta t}{\varepsilon^2} \quad \overset{\Delta t \to 0}{\to} \quad 0$$

This proves the theorem. ■

We summarize our results: The statistics of financial data suggests, as a first approximation, the stochastic model

$$S_{t_k} \quad = \quad S_{t_{k-1}} (1 + \mu \, \Delta t^{\alpha} + \sigma \, \Delta t^{\beta} \phi_k)$$

with $\alpha = 1$ and $\beta = \frac{1}{2}$ which results in

$$\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \;=\; \frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\,\Delta t + \sigma\,\sqrt{\Delta t}\,\phi_k \tag{2}$$

Since a Brownian motion was defined as the combination of random numbers

$$x_{t_k} \;=\; \sqrt{\Delta t}\,\sum_{j=1}^{k}\phi_j$$

we can write

$$\sqrt{\Delta t}\,\phi_k \;=\; x_{t_k} \;-\; x_{t_{k-1}}$$

which results in

$$\frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\,\Delta t + \sigma\,(x_{t_k} - x_{t_{k-1}}) \;=\; \mu\,\Delta t \;+\; \sigma\,\Delta x_{t_k} \tag{3}$$

or, in the continuous time limit $\Delta t \to 0$,

$$\frac{dS_t}{S_t} \;=\; \mu\,dt \;+\; \sigma\,dx_t \tag{4}$$

where $\{x_t\}_{0<t\leq T}$ is a Brownian motion. And we saw that the discrete time solution (1) of (3) (for $\mu \neq 0$ there is an additional $\mu N_t \Delta t$ in the exponent, as we had in the previous lecture) converges to

$$S_t \;=\; S_0\,e^{\mu t + \sigma x_t - \frac{\sigma^2}{2}t} \tag{5}$$

The solution (5) is usually refered to as a 'geometric Brownian motion'. In a following section we will rederive (5) from (4) as an application of the Ito-Lemma. We have the following very important

Definition 4.2: Let $\{x_t\}_{t\geq 0}$ be a Brownian motion. Then the stochastic process (5),

$$S_t \;=\; S_0\,e^{\mu t + \sigma x_t - \frac{\sigma^2}{2}t} \tag{6}$$

is called the **Black-Scholes model** for the asset price process $\{S_t\}_{t\geq 0}$. It is a solution of the stochastic differential equation (4),

$$\tfrac{dS_t}{S_t} \;=\; \mu\,dt \;+\; \sigma\,dx_t \tag{7}$$

Equation (7) is called the **Black-Scholes Stochastic Differential Equation** or Black-Scholes SDE (not to be confused with the Black-Scholes PDE, partial differential equation).

In discrete time, Black-Scholes paths can be simulated through

$$S_{t_k} \;=\; S_{t_{k-1}}\,\big(\,1 \;+\; \mu\,\Delta t \;+\; \sigma\,\sqrt{\Delta t}\,\phi_k\,\big) \tag{8}$$

with the ϕ_k being standard normal distributed random numbers.

Excel/VBA-Demos:

a) Show through simulation that for small Δt the S_{t_k}'s calculated iteratively through (8) and directly through (6) are approximately equal.

b) Confirm part (c) of Theorem 4.2. That is, show through simulation that in discrete time for small Δt

$$I_{\Delta t} \;:=\; \Delta t \sum_{k=1}^{N_t} \phi_k^2 \;\approx\; t \;.$$