week6: Kapitel 4: Brownsche Bewegung, Wiener-Maß und das Black-Scholes Modell, Teil1

Consider some discrete times t_k in the intervall $[0, T]$,

$$t_k \;=\; k\tfrac{T}{N} \;=\; k\Delta t\,, \qquad k = 0, 1, ..., N \tag{1}$$

where

$$N = N_T = \tfrac{T}{\Delta t} \in \mathbb{N} \tag{2}$$

Let $S_{t_k} = S_{k\Delta t}$ be the price of some stock at time t_k and denote the returns by going from one time step to the next by

$$\mathrm{ret}_{t_k} \;=\; \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \tag{3}$$

One may think of Δt being one day and S_{t_k} being the closing prices at each day although later we will consider the limit $\Delta t \to 0$. It is an empirical fact that the daily returns of many assets are bell shaped, like a Gaussian distribution. Letzte Woche hatten wir uns dazu die DAX-Returns der letzten 35 Jahre, von 1988 bis heute, angeschaut. Die Schlussfolgerung war dann, das hatten wir auch am letzten Freitag gemacht: As a first approximation, one may write down the following stochastic model for the returns:

$$\mathrm{ret}_{t_k} \;=\; \text{mean} \;+\; \text{standard deviation} \times \phi_k \tag{4}$$

where the ϕ_k are identically independent normally distributed random numbers with mean zero and variance one,

$$\phi_k \in \mathcal{N}(0,1) \quad \text{i.i.d.} \tag{5}$$

This is only a first approximation. There are deviations from a Gaussian distribution. Most financial data have more heavy tails than a normal distribution and a higher peak at the mean value. Furthermore, the returns in (4) are not completely independent. Many financial data show a positive correlation of the absolute values of the returns, of $|\mathrm{ret}_{t_k}|$ and $|\mathrm{ret}_{t_{k+m}}|$. In the book of Shiryaev *Essentials of Stochastic Finance* one can find a detailed discussion of the statistical analysis of financial data (in Chapter 4) as well as an overview of the proposed stochastic models to fit these data.

We now analyze how the mean and the standard deviation in (4) have to scale with Δt in order to get a reasonable model in the time continuous case $\Delta t \to 0$. To this end we write

$$\mathrm{ret}_{t_k} \;=\; \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\,\phi_k \tag{6}$$

such that

$$S_{t_k} = S_{t_{k-1}}(1 + \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\phi_k)$$

or, with $t = N_t \times \Delta t$, $N_t = t/\Delta t$,

$$S_t = S_0 \prod_{k=1}^{N_t} \left(1 + \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\phi_k\right) \tag{7}$$

Suppose for the moment the model is deterministic, $\sigma = 0$. Then, using the first order Taylor expansion $\log(1+x) = x + O(x^2)$ in the third line,

$$\begin{aligned} S_t &= S_0\,(1 + \mu\,\Delta t^{\alpha})^{N_t} \\ &= S_0\,e^{N_t \log(1+\mu\,\Delta t^{\alpha})} \\ &= S_0\,e^{N_t(\mu\Delta t^{\alpha} + O(\Delta t^{2\alpha}))} \\ &= S_0\,e^{\mu t \Delta t^{\alpha-1} + O(\Delta t^{2\alpha-1})} \end{aligned} \tag{8}$$

which gives $\alpha = 1$ and exponential growth (or decrease) in the time continuous case, $S_t = S_0\,e^{\mu t}$ which is simply the solution of $dS/S = \mu dt$. Now consider the stochastic part in (6). For simplicity, we put $\mu = 0$. Then, now using the second order Taylor expansion $\log(1+x) = x - x^2/2 + O(x^3)$ in the third line,

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} \left(1 + \sigma\,\Delta t^{\beta}\phi_k\right) \\ &= S_0\,e^{\sum_{k=1}^{N_t} \log(1+\sigma\,\Delta t^{\beta}\phi_k)} \\ &= S_0\,e^{\sum_{k=1}^{N_t}\left(\sigma\,\Delta t^{\beta}\phi_k - \frac{1}{2}\sigma^2\Delta t^{2\beta}\phi_k^2 + O(\Delta t^{3\beta})\right)} \\ &= S_0\,e^{\sigma\,\Delta t^{\beta}\sum_{k=1}^{N_t}\phi_k - \frac{\sigma^2}{2}\Delta t^{2\beta}\sum_{k=1}^{N_t}\phi_k^2 + O(N_t\Delta t^{3\beta} = \Delta t^{3\beta-1})} \end{aligned} \tag{9}$$

We now consider for what values of β the expectation

$$\mathsf{E}\left[f\left(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\right)\right] = \int_{\mathbb{R}^{N_t}} f\left(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\right) \prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\phi_k^2}{2}}\,d\phi_k \tag{10}$$

has a nontrivial limit. Here f is some function. We make a substitution of variables $(\phi_k)_{1\le k\le N_t} \to (x_k)_{1\le k\le N_t}$ defined as follows:

$$\begin{array}{lcl c lcl} x_1 &=& \sqrt{\Delta t}\,\phi_1 & & \phi_1 &=& x_1/\sqrt{\Delta t} \\ x_2 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2) & & \phi_2 &=& (x_2-x_1)/\sqrt{\Delta t} \\ x_3 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+\phi_3) & \Leftrightarrow & \phi_3 &=& (x_3-x_2)/\sqrt{\Delta t} \\ \vdots & & & & \vdots & & \\ x_{N_t} &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+\ldots+\phi_{N_t}) & & \phi_{N_t} &=& (x_{N_t}-x_{N_t-1})/\sqrt{\Delta t} \end{array} \tag{11}$$

The Jacobian of the transformation (11) is $\det\frac{\partial\phi}{\partial x} = 1/\sqrt{\Delta t}^{N_t}$ since, with $N = N_t$,

$$\frac{\partial\phi}{\partial x} = \begin{pmatrix} - & \nabla_x \phi_1 & - \\ & \vdots & \\ - & \nabla_x \phi_N & - \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_N} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\phi_N}{\partial x_1} & \frac{\partial\phi_N}{\partial x_2} & \cdots & \frac{\partial\phi_N}{\partial x_N} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} \end{pmatrix}$$

Thus the expectation (10) becomes, with $N = N_t = t/\Delta t$,

$$\begin{aligned} \mathsf{E}\Big[f\Big(\Delta t^\beta \textstyle\sum_{k=1}^N \phi_k\Big)\Big] &= \int_{\mathbb{R}^N} f\Big(\Delta t^\beta \textstyle\sum_{k=1}^{N_t} \phi_k\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}}\, d\phi_k \Big\} \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}} \sqrt{\Delta t} \textstyle\sum_{k=1}^N \phi_k\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}} \Big\}\, d\phi_1 \cdots d\phi_N \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}}\, x_N\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \Big\} \det\big[\tfrac{\partial\phi}{\partial x}\big]\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}}\, x_N\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \Big\} \frac{1}{(\sqrt{\Delta t})^N}\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}}\, x_N\Big) \prod_{k=1}^N \Big\{ \frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(x_{k-1} - x_k)^2}{2\Delta t}} \Big\}\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_N\big) \prod_{k=1}^N \Big\{ p_{\Delta t}(x_{k-1}, x_k)\, dx_k \Big\} \end{aligned} \tag{12}$$

where we introduced the kernel

$$p_\tau(x,y) := \frac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \tag{13}$$

and used the definition

$$x_0 := 0 \tag{14}$$

The kernel (13) has the following basic property:

Lemma 4.1: *Let $p_t(x,y)$ be given by (13). Then*

$$\int_{\mathbb{R}} p_s(x,y) p_t(y,z)\, dy = p_{s+t}(x,z) \tag{15}$$

Proof: Übungsblatt 6. ■

Using this lemma, we can actually perform the integrals over $x_1, x_2, \cdots, x_{N-1}$. We have

$$\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \cdots \int_{\mathbb{R}} dx_{N-1}\; \underbrace{p_{\Delta t}(x_0,x_1)\, p_{\Delta t}(x_1,x_2)}_{\int dx_1 \,\to\, p_{2\Delta t}(x_0,x_2)}\; p_{\Delta t}(x_2,x_3)\; \cdots\; p_{\Delta t}(x_{N-1},x_N)$$

$$= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1}\; \underbrace{p_{2\Delta t}(x_0,x_2)\, p_{\Delta t}(x_2,x_3)}_{\int dx_2 \,\to\, p_{3\Delta t}(x_0,x_3)}\; \cdots\; p_{\Delta t}(x_{N-1},x_N)$$

$$= \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1}\; p_{3\Delta t}(x_0,x_3)\; \cdots\; p_{\Delta t}(x_{N-1},x_N)$$

$$= \int_{\mathbb{R}} dx_{N-1}\; p_{(N-1)\Delta t}(x_0,x_{N-1})\, p_{\Delta t}(x_{N-1},x_N)$$

$$= p_{N\Delta t}(x_0,x_N)$$

Thus (12) simplifies to

$$\mathsf{E}\Big[f\Big(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\Big)\Big] \;=\; \int_{\mathbb{R}^N} f\big(\Delta t^{\beta-\frac{1}{2}}\,x_N\big)\prod_{k=1}^{N}\Big\{\,p_{\Delta t}(x_{k-1},x_k)\,dx_k\,\Big\} \tag{16}$$

$$= \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\,x_N\big) \;\times\; p_{N\Delta t}(x_0,x_N)\,dx_N$$

$$\overset{\substack{N\Delta t = t\\ x_0=0}}{=} \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\,x\big) \;\times\; \tfrac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}\,dx \tag{17}$$

Hence, a nontrivial meaningful limit is obtained only for $\beta = \frac{1}{2}$.

Instead of labelling the x with $k \in \{1,2,...,N_t\}$, we label them with $t_k := k\Delta t$ which has the meaning of time. In particular, $t_N = N\Delta t = t$. So, we rename $x_k \to x_{k\Delta t} = x_{t_k}$. With that, we write down the following very important

Definition 4.1: *Let $N_T = T/\Delta t$ and $p_t(x,y) := \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(x-y)^2}{2t}}$. Then the measure*

$$dW(\{x_t\}_{0<t\le T}) \;:=\; \lim_{\Delta t\to 0}\prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\,dx_{k\Delta t} \tag{18}$$

is called the Wiener measure and the family of random variables or integration variables $\{x_t\}_{0<t\le T}$ is called a Brownian motion. In terms of i.i.d. random variables $\phi_k \in \mathcal{N}(0,1)$,

$$x_t \;:=\; \lim_{\Delta t\to 0}\sqrt{\Delta t}\sum_{k=1}^{t/\Delta t}\phi_k \tag{19}$$

Remark: The time discretized version of the Wiener measure, this is what we actually will usually use, is simply given by a product of Gaussian normal distributions. With $N = N_T = T/\Delta t$ and $t_k = k\Delta t$,

$$dW(\{x_{t_k}\}_{0<k\le N}) \;=\; \prod_{k=1}^{N} p_{\Delta t}(x_{t_{k-1}}, x_{t_k})\,dx_{t_k}$$

$$= \prod_{k=1}^{N}\Big\{\,e^{-\frac{(x_{t_k}-x_{t_{k-1}})^2}{2\Delta t}}\;\frac{dx_{t_k}}{\sqrt{2\pi\Delta t}}\,\Big\}$$

$$= \prod_{k=1}^{N}\Big\{\,e^{-\frac{\phi_k^2}{2}}\;\frac{d\phi_k}{\sqrt{2\pi}}\,\Big\} \tag{20}$$

And the time discretized Brownian motion x_{t_k} is given by

$$x_{t_k} = \sqrt{\Delta t} \sum_{j=1}^{k} \phi_j \tag{21}$$

from which we get the recursion

$$x_{t_k} = x_{t_{k-1}} + \sqrt{\Delta t}\, \phi_k \tag{22}$$

The formulae (20,21,22) are very important and will be used over and over again.