week2b: Kapitel 2: Das Binomialmodell, Teil2

Letztes Mal hatten wir das Binmialmodell als ein zeitdiskretes Assetpreismodell definiert, was von einem Zeitpunkt t_{k-1} zum nächsten Zeitpunkt t_k jeweils immer nur 2 Einstellungmöglichkeiten zulässt, das war die

Definition 2.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$S_k \;=\; S_{k-1}(\,1\,+\,\mathrm{ret}_k\,) \qquad \text{with} \quad \mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\} \tag{1}$$

for all k, then we say that S is given by the Binomial model.

Dann hatten wir uns noch kurz das folgende Theorem angeschaut, was wir jetzt also beweisen wollen:

Theorem 2.1: Let S be some tradable asset whose price process is given by the Binomial model (1). Let r denote some constant interest rate which is to be applied to cash amounts at each period from t_{k-1} to t_k. Then every option payoff

$$H \;=\; H(S_0, ..., S_N)$$

can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \delta_k(S_0, \cdots, S_k) \;=\; \frac{V_{k+1}^{\mathrm{up}} - V_{k+1}^{\mathrm{down}}}{S_{k+1}^{\mathrm{up}} - S_{k+1}^{\mathrm{down}}} \tag{2}$$

with the abbreviations

$$\begin{aligned} S_{k+1}^{\mathrm{up/down}} &:= S_k\,(\,1\,+\,\mathrm{ret}_{\mathrm{up/down}}\,) \\ V_{k+1}^{\mathrm{up/down}} &:= V_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\mathrm{up/down}}\,\big) \end{aligned}$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae: Let

$$R \;:=\; 1 + r$$

Then

$$V_k \;=\; R^k\, v_k$$

and the discounted portfolio values v_k can be calculated recursively through

$$v_k \;=\; w_{\rm up}\, v_{k+1}^{\rm up} \;+\; w_{\rm down}\, v_{k+1}^{\rm down} \tag{3}$$

and the recursion starts at $k = N$ with discounted portfolio values

$$v_N \;:=\; R^{-N}\, H(S_0, \cdots, S_N)$$

The weights $w_{\rm up}$ and $w_{\rm down}$ are given by

$$w_{\rm up} \;=\; \frac{r \,-\, {\rm ret}_{\rm down}}{{\rm ret}_{\rm up} \,-\, {\rm ret}_{\rm down}} \tag{4}$$

$$w_{\rm down} \;=\; 1 - w_{\rm up} \;=\; \frac{{\rm ret}_{\rm up} \,-\, r}{{\rm ret}_{\rm up} \,-\, {\rm ret}_{\rm down}} \;. \tag{5}$$

Proof of Theorem 2.1: For nonzero interest rates we have

$$v_N \;=\; v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$

which is equivalent to

$$v_{k+1} \;=\; v_k + \delta_k(s_{k+1} - s_k) \qquad \forall\; k = 0, 1, ..., N-1 \tag{6}$$

We have

$$v_{k+1} \;=\; v_{k+1}(S_0, \cdots, S_k, S_{k+1}) \;=\; v_{k+1}\big(\, S_0\,,\; \cdots\,,\; S_k\,,\; S_k(1 + {\rm ret}_k)\,\big)$$

and the return ${\rm ret}_k$ can be an up-move or a down-move in which case we get

$$\begin{aligned} v_{k+1}^{\rm up} \;&=\; v_{k+1}\big(\, S_0\,,\; \cdots\,,\; S_k\,,\; S_{k+1}^{\rm up}\,\big) \;=\; v_{k+1}\big(\, S_0\,,\; \cdots\,,\; S_k\,,\; S_k(1 + {\rm ret}_{\rm up})\,\big) \\ v_{k+1}^{\rm down} \;&=\; v_{k+1}\big(\, S_0\,,\; \cdots\,,\; S_k\,,\; S_{k+1}^{\rm down}\,\big) \;=\; v_{k+1}\big(\, S_0\,,\; \cdots\,,\; S_k\,,\; S_k(1 + {\rm ret}_{\rm down})\,\big) \end{aligned}$$

From (6), we have

$$\begin{aligned} v_{k+1}^{\rm up} \;&=\; v_k + \delta_k(s_{k+1}^{\rm up} - s_k) \\ v_{k+1}^{\rm down} \;&=\; v_k + \delta_k(s_{k+1}^{\rm down} - s_k) \end{aligned}$$

Thus,

$$v_{k+1}^{\rm up} \,-\, v_{k+1}^{\rm down} \;=\; \delta_k(s_{k+1}^{\rm up} \,-\, s_{k+1}^{\rm down})$$

or

$$\delta_k \;=\; \frac{v_{k+1}^{\rm up} \,-\, v_{k+1}^{\rm down}}{s_{k+1}^{\rm up} \,-\, s_{k+1}^{\rm down}} \;=\; \frac{R^{-(k+1)}(V_{k+1}^{\rm up} \,-\, V_{k+1}^{\rm down})}{R^{-(k+1)}(S_{k+1}^{\rm up} \,-\, S_{k+1}^{\rm down})} \;=\; \frac{V_{k+1}^{\rm up} \,-\, V_{k+1}^{\rm down}}{S_{k+1}^{\rm up} \,-\, S_{k+1}^{\rm down}}$$

Solving (6) for v_k,

$$\begin{aligned} v_k &= v_{k+1} - \delta_k(s_{k+1} - s_k) \\ &= v_{k+1}^{\text{up}} - \delta_k(s_{k+1}^{\text{up}} - s_k) \\ &= v_{k+1}^{\text{down}} - \delta_k(s_{k+1}^{\text{down}} - s_k) \end{aligned}$$

Let's take the up-equation and substitute the value for δ_k (we also could use the down-equation, we would end up with the same result),

$$\begin{aligned} v_k &= v_{k+1}^{\text{up}} - \delta_k(s_{k+1}^{\text{up}} - s_k) \\ &= v_{k+1}^{\text{up}} - \frac{v_{k+1}^{\text{up}} - v_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}(s_{k+1}^{\text{up}} - s_k) \\ &= \frac{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} v_{k+1}^{\text{up}} - (v_{k+1}^{\text{up}} - v_{k+1}^{\text{down}}) \frac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} \\ &= \frac{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}} - s_{k+1}^{\text{up}} + s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} v_{k+1}^{\text{up}} + \frac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} v_{k+1}^{\text{down}} \\ &= \frac{s_k - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} v_{k+1}^{\text{up}} + \frac{s_{k+1}^{\text{up}} - s_k}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} v_{k+1}^{\text{down}} \\ &=: w_{\text{up}}\, v_{k+1}^{\text{up}} + w_{\text{down}}\, v_{k+1}^{\text{down}} \end{aligned}$$

with weights w_{up} and w_{down} which apparently add up to 1 and

$$w_{\text{up}} = \frac{s_k - s_{k+1}^{\text{down}}}{s_{k+1}^{\text{up}} - s_{k+1}^{\text{down}}} = \frac{R^{-k}S_k - R^{-(k+1)}S_{k+1}^{\text{down}}}{R^{-(k+1)}(S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}})} = \frac{RS_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}}$$

or

$$\begin{aligned} w_{\text{up}} &= \frac{RS_k - S_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} \\ &= \frac{(1+r)S_k - S_k(1+\text{ret}_{\text{down}})}{S_k(1+\text{ret}_{\text{up}}) - S_k(1+\text{ret}_{\text{down}})} \\ &= \frac{1+r - (1+\text{ret}_{\text{down}})}{(1+\text{ret}_{\text{up}}) - (1+\text{ret}_{\text{down}})} \\ &= \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \end{aligned}$$

and the theorem is proven. ■

Remarks: 1) If H is a path-independent or non-exotic option which depends only on the underlying price at maturity,

$$H = H(S_N)$$

then the δ_k and the value of the replicating portfolio V_k at t_k depend only on the asset price S_k and do not depend on earlier prices $S_{k-1}, S_{k-2}, ..., S_0$. That is,

$$\begin{aligned} V_k &= V_k(S_k) \\ \delta_k &= \delta_k(S_k) \end{aligned}$$

2) Assume zero interest rates $r = 0$ such that $R = 1 + r = 1$. Then $v_k = V_k$ and (3) becomes

$$V_k \;=\; w_{\text{up}}\, V_{k+1}^{\text{up}} \;+\; w_{\text{down}}\, V_{k+1}^{\text{down}}$$

with weights

$$w_{\text{up}} \;=\; \frac{r \;-\; \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \;-\; \text{ret}_{\text{down}}} \;=\; \frac{-\;\text{ret}_{\text{down}}}{\text{ret}_{\text{up}} \;-\; \text{ret}_{\text{down}}}$$

$$w_{\text{down}} \;=\; \frac{+\;\text{ret}_{\text{up}}}{\text{ret}_{\text{up}} \;-\; \text{ret}_{\text{down}}}$$

If we further assume a 'symmetric' Binomial model with (say, $q = 1\%$ or $q = 5\%$)

$$\text{ret}_{\text{up}} \;=\; +\,q$$
$$\text{ret}_{\text{down}} \;=\; -\,q$$

the weights simplify to

$$w_{\text{up}} \;=\; \frac{-(-q)}{2q} \;=\; \frac{1}{2} \;=\; w_{\text{down}}$$

and we arrive at the simple recursion formula

$$V_k \;=\; \frac{V_{k+1}^{\text{up}} \;+\; V_{k+1}^{\text{down}}}{2} \tag{7}$$

Schauen wir uns jetzt ein konkretes Beispiel dazu an, das machen wir an der Tafel.