

## Kapitel 6: Die Black-Scholes Formeln und Options-Sensitivitäten ('Greeks')

Wir benutzen jetzt die grundlegende Pricing-Formel (5) aus der 2. Aufgabe vom 10. Übungsblatt,

$$V_0^{\text{BS}} = e^{-rT} \int_{\mathbb{R}} H\left(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x}\right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

um die doch ziemlich berühmten Black-Scholes Formeln herzuleiten. Dann definieren und berechnen wir noch ein paar 'greeks', das sind einfach Ableitungen des Optionspreises nach den verschiedenen Parametern wie  $S$ ,  $\sigma$ ,  $r$  oder  $T$ :

**Definition 6.1:** Let  $H = H(\{S_t\}_{0 \leq t \leq T})$  be the payoff of some (possibly path dependent) option and let

$$V_t = V(S_t, \sigma, r, t) \tag{1}$$

denote the time  $t$  theoretical fair value of  $H$  (evaluated with respect to some asset price model, not necessarily the Black-Scholes model). Here  $S_t$  is the asset price at time  $t$ ,  $\sigma$  denotes the volatility and  $r$  denotes the interest rates. Then the following abbreviations has been become standard in the derivatives community (listed with decreasing importance):

$$\delta := \frac{\partial V_t}{\partial S_t} \quad (\text{delta}) \tag{2}$$

$$vega := \frac{\partial V_t}{\partial \sigma} \quad (\text{vega}) \tag{3}$$

$$\rho := \frac{\partial V_t}{\partial r} \quad (\text{rho}) \tag{4}$$

$$\theta := \frac{\partial V_t}{\partial t} \quad (\text{theta}) \tag{5}$$

$$\gamma := \frac{\partial^2 V_t}{\partial S_t^2} \quad (\text{gamma}) \tag{6}$$

**Theorem 6.1 (Black-Scholes Formeln):** Consider a call and a put option with strike  $K$  and maturity  $T$ ,

$$H_{\text{call}} = \max\{S_T - K, 0\} \tag{7}$$

$$H_{\text{put}} = \max\{K - S_T, 0\} \tag{8}$$

and let  $V_{\text{call},t}^{\text{BS}}$  and  $V_{\text{put},t}^{\text{BS}}$  denote their time  $t$  theoretical fair values in the Black-Scholes model. Let

$$N(x) := \int_{-\infty}^x e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \tag{9}$$

denote the cumulative normal distribution function and define the quantities

$$d_{\pm} := \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad (10)$$

such that  $d_+ - d_- = \sigma \sqrt{T-t}$ . Then there are the following formulae:

**Fair Value:**

$$V_{\text{call},t}^{\text{BS}} = S_t N(d_+) - K e^{-r(T-t)} N(d_-) \quad (11)$$

$$V_{\text{put},t}^{\text{BS}} = -S_t N(-d_+) + K e^{-r(T-t)} N(-d_-) \quad (12)$$

**Delta:**

$$\delta_{\text{call}} = N(d_+) \quad (13)$$

$$\delta_{\text{put}} = -N(-d_+) \quad (14)$$

**Vega:**

$$vega_{\text{call}} = S_t N'(d_+) \sqrt{T-t} \quad (15)$$

$$vega_{\text{put}} = vega_{\text{call}} \quad (16)$$

**Rho:**

$$\rho_{\text{call}} = K(T-t) e^{-r(T-t)} N(d_-) \quad (17)$$

$$\rho_{\text{put}} = -K(T-t) e^{-r(T-t)} N(-d_-) \quad (18)$$

**Theta:**

$$\theta_{\text{call}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(d_+) - r K e^{-r(T-t)} N(d_-) \quad (19)$$

$$\theta_{\text{put}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(-d_+) + r K e^{-r(T-t)} N(-d_-) \quad (20)$$

**Gamma:**

$$\gamma_{\text{call}} = \frac{N'(d_+)}{\sigma \sqrt{T-t} S_t} \quad (21)$$

$$\gamma_{\text{put}} = \gamma_{\text{call}} \quad (22)$$

**Proof:** In the Black-Scholes model, the time  $t$  theoretical fair value of some european option with maturity  $T \geq t$  and payoff  $H = H(S_T)$  is given by (in Übungsblatt 10 we solved the Black-Scholes PDE)

$$V_t^{\text{BS}} = e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma \sqrt{T-t} x + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad (23)$$

Let us abbreviate

$$\tau := T - t \quad (24)$$

$$a := \sigma \sqrt{\tau} \quad (25)$$

$$b := (r - \sigma^2/2)\tau \quad (26)$$

and define  $x_0$  as the solution of the equation

$$S_t e^{\sigma\sqrt{T-t}x + (r - \frac{\sigma^2}{2})(T-t)} = S_t e^{ax+b} \stackrel{!}{=} K \quad (27)$$

That is,

$$x_0 = \frac{1}{a} \log(K/S_t) - \frac{b}{a} \quad (28)$$

Then

$$\begin{aligned} V_{\text{call},t}^{\text{BS}} &= e^{-r\tau} \int_{\mathbb{R}} H_{\text{call}}(S_t e^{ax+b}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{\mathbb{R}} \max\{S_t e^{ax+b} - K, 0\} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} S_t e^{b + \frac{a^2}{2}} \int_{x_0}^{\infty} e^{-\frac{1}{2}(x-a)^2} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} K \int_{x_0}^{\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r\tau} S_t e^{b + \frac{a^2}{2}} N(-x_0 + a) - e^{-r\tau} K N(-x_0) \\ &= S_t N(d_+) - e^{-r(T-t)} K N(d_-) \end{aligned} \quad (29)$$

since

$$-x_0 = \frac{1}{a} \log(S_t/K) + \frac{b}{a} = \frac{\log(S_t/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_-$$

and  $-x_0 + a = d_- + \sigma\sqrt{\tau} = d_+$ . Furthermore

$$-r\tau + b + a^2/2 = -r\tau + (r - \sigma^2/2)\tau + (\sigma\sqrt{\tau})^2/2 = 0$$

which results in the last line of (29). The proof for the put fair value is similar. In particular, since

$$N(d) + N(-d) = 1$$

we get from (11,12)

$$V_{\text{call},t}^{\text{BS}} - V_{\text{put},t}^{\text{BS}} = S_t - e^{-r(T-t)} K \quad (30)$$

This relation is called ‘call-put parity’.

**Delta:** To compute the delta we use the third line of (29) instead of the last one. We get

$$\begin{aligned} \delta_{\text{call}}(S_t, t) &= \frac{\partial}{\partial S_t} \left\{ e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right\} \\ &= e^{-r\tau} \int_{x_0}^{\infty} e^{ax+b} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} \underbrace{(S_t e^{ax_0+b} - K)}_{=0} \frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}} \frac{\partial x_0}{\partial S_t} \\ &= N(d_+) \end{aligned} \quad (31)$$

since the above integral is the same as the first integral in (29), up to the factor of  $S_t$  which has been differentiated away. Finally it follows from call-put parity that

$$\delta_{\text{put}}(S, t) = N(d_+) - 1 = -N(-d_+).$$

**Vega:** Again we use the third line of (29):

$$\begin{aligned}
vega_{\text{call}}(S_t, t) &= \frac{\partial}{\partial \sigma} \left\{ e^{-r\tau} \int_{x_0}^{\infty} (S_t e^{ax+b} - K) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right\} \\
&= e^{-r\tau} \int_{x_0}^{\infty} S_t \left( \frac{\partial a}{\partial \sigma} x + \frac{\partial b}{\partial \sigma} \right) e^{ax+b} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - e^{-r\tau} \underbrace{(S_t e^{ax_0+b} - K)}_{=0} \frac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}} \frac{\partial x_0}{\partial \sigma} \\
&= e^{-r\tau} e^{b+\frac{a^2}{2}} S_t \int_{x_0}^{\infty} (\sqrt{\tau}x - \sigma\tau) e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= S_t \sqrt{\tau} \int_{x_0}^{\infty} (x-a) e^{-\frac{(x-a)^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= S_t \sqrt{\tau} \left[ -\frac{e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}} \right]_{x_0}^{\infty} = S_t \sqrt{\tau} \frac{e^{-\frac{(x_0-a)^2}{2}}}{\sqrt{2\pi}} \\
&= S_t \sqrt{\tau} \frac{e^{-\frac{(-d_+)^2}{2}}}{\sqrt{2\pi}} = S_t \sqrt{\tau} \frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} = S_t \sqrt{\tau} N'(d_+) \tag{32}
\end{aligned}$$

The vega for the put follows again from call-put parity. The remaining formulae may be looked up, for example, in the book on Quantitative Finance, Vol.1, by Paul Wilmott. ■

Schreiben wir hier noch einmal die allgemeine Pricing-Formel für den Fall beliebiger Zeiten  $t \in [0, T]$  hin,

$$V_t^{\text{BS}} = e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma\sqrt{T-t}x + (r-\frac{\sigma^2}{2})(T-t)}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \tag{33}$$

wenn  $H = H(S_T)$  die Optionsauszahlung einer beliebigen pfadunabhängigen Option ist. Für Zeit  $t = 0$  reduziert sich das dann also auf

$$V_0^{\text{BS}} = e^{-rT} \int_{\mathbb{R}} H(S_0 e^{\sigma\sqrt{T}x + (r-\frac{\sigma^2}{2})T}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \tag{34}$$

und für Zeit  $t = T$ , Fälligkeit der Option,

$$\begin{aligned}
V_T^{\text{BS}} &= e^{-r \cdot 0} \int_{\mathbb{R}} H(S_T e^{\sigma\sqrt{0}x + (r-\frac{\sigma^2}{2}) \cdot 0}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= \int_{\mathbb{R}} H(S_T) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= H(S_T) \quad .
\end{aligned}$$