week11b: Kapitel 5.3: Die Black-Scholes PDE und Payoff-Replication, Teil2

Nach den Vorbereitungen vom letzten Mal können wir jetzt das folgende konzeptionell sehr wichtige Theorem 5.3.4 formulieren, das aussagt, dass im Black-Scholes Modell immer noch jede (europäische, pfadunabhängige) Option durch eine geeignete Handelsstrategie repliziert werden kann:

Theorem 5.3.4: Let $H = H(S)$ be some option payoff and let $V = V(S,t)$ be a solution of the Black-Scholes PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \\ V(S,T) &= H(S) \end{aligned}$$

and let $\{S_t\}_{t\geq 0}$ be a stochastic price process given by the Black-Scholes Modell with real world drift μ,

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dx_t$$

Furthermore let $V_0 = V(S_0, 0)$ be the option price of H and define a trading strategy by the following delta's,

$$\delta_t = \delta(S_t, t) := \frac{\partial V(S_t,t)}{\partial S_t}$$

As usual, we use the notation with small letters

$$s_t := e^{-rt} S_t$$

for the discounted price process. Then the following equation holds:

$$e^{-rT} H(S_T) = V_0 + \int_0^T \delta_t\, ds_t$$

This equation means (recall part (b) of Theorem 1.1 in the first chapter) that in the Black-Scholes model any option payoff $H = H(S_T)$ can be replicated with a suitable trading strategy defined by the delta's above.

Proof: Let

$$v(S,t) := e^{-rt}\, V(S,t)$$

Then, since $v_0 = V_0$,

$$e^{-rT} H(S_T) \;-\; V_0 \;=\; v(S_T,T) \;-\; v(S_0,0) \;=\; \int_0^T dv$$

where

$$\begin{aligned} dv &= d\big(e^{-rt} V\big) \\ &= d(e^{-rt})\,V \;+\; e^{-rt}\,dV \;+\; d(e^{-rt})\,dV \\ &= -r\,e^{-rt}\,dt\,V \;+\; e^{-rt}\,dV \;+\; 0 \end{aligned}$$

Now,

$$dV \;=\; \frac{\partial V}{\partial S}\,dS_t \;+\; \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2 \;+\; \frac{\partial V}{\partial t}\,dt$$

Using

$$\delta_t \;=\; \frac{\partial V}{\partial S}(S_t,t)$$

and, recalling the calculation rules for Brownian motion $(dx_t)^2 = dt$ and $dx_t\,dt = (dt)^2 = 0$,

$$\begin{aligned} (dS_t)^2 &= (\mu\,S_t\,dt + \sigma\,S_t\,dx_t)^2 \\ &= 0 + 0 + \sigma^2\,S_t^2\,(dx_t)^2 \\ &= \sigma^2\,S_t^2\,dt \end{aligned}$$

we obtain

$$dV \;=\; \delta_t\,dS_t \;+\; \Big\{\frac{\sigma^2}{2} S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\Big\}\,dt$$

We have to express dS_t through ds_t where $s_t = e^{-rt}\,S_t$. We have

$$ds_t \;=\; d(e^{-rt} S_t) \;=\; -r\,e^{-rt}\,dt\,S_t \;+\; e^{-rt}\,dS_t$$

or

$$e^{-rt}\,dS_t \;=\; ds_t \;+\; r\,e^{-rt}\,dt\,S_t$$

which gives

$$\begin{aligned} e^{-rt}\,dV &= \delta_t\,e^{-rt}\,dS_t \;+\; e^{-rt}\Big\{\frac{\sigma^2}{2} S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\Big\}\,dt \\ &= \delta_t\,\big[ds_t \;+\; r\,e^{-rt}\,dt\,S_t\big] \;+\; e^{-rt}\Big\{\frac{\sigma^2}{2} S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\Big\}\,dt \\ &= \delta_t\,ds_t \;+\; rS_t\frac{\partial V}{\partial S}\,dt \;+\; e^{-rt}\Big\{\frac{\sigma^2}{2} S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\Big\}\,dt \end{aligned}$$

Thus,

$$
\begin{aligned}
dv &= -r\,e^{-rt}\,dt\,V \;+\; e^{-rt}\,dV \\
&= -r\,e^{-rt}\,dt\,V \;+\; \delta_t\,ds_t \;+\; rS_t\frac{\partial V}{\partial S}\,dt \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; \frac{\partial V}{\partial t}\right\}dt \\
&= \delta_t\,ds_t \;+\; e^{-rt}\left\{\frac{\sigma^2}{2}S_t^2\,\frac{\partial^2 V}{\partial S^2} \;+\; rS_t\frac{\partial V}{\partial S} \;+\; \frac{\partial V}{\partial t} \;-\; rV\right\}dt \\
&= \delta_t\,ds_t
\end{aligned}
$$

where we used the Black-Scholes PDE in the last line. Thus,

$$
\begin{aligned}
e^{-rT}H(S_T) \;-\; V_0 &= v(S_T, T) \;-\; v(S_0, 0) \\
&= \int_0^T dv \\
&= \int_0^T \delta_t\,ds_t
\end{aligned}
$$

and the theorem is proven. ■