week11a: Kapitel 5.3: Die Black-Scholes PDE, Teil1

In the the Black-Scholes model, the price of some liquid tradable asset or option underlying S_t is modelled through the stochastic differential equation

$$dS_t/S_t \;=\; \mu\, dt + \sigma\, dx_t \tag{1}$$

If the underlying price dynamics is modelled by the Binomial model, we know that every option payoff $H = H(S_T)$ can be replicated by a suitable trading strategy in the underling. A crucial point in the derivation of that result was, that from one time step to the next, there are only two possible choices for $S(t_k)$ given the value of $S(t_{k-1})$. In the Black-Scholes model, the discretized price dynamics is given by

$$S_{t_k} \;=\; S_{t_{k-1}}\,(\,1 \,+\, \mu\,\Delta t \,+\, \sigma\sqrt{\Delta t}\;\phi_k\,) \tag{2}$$

and, given $S_{t_{k-1}}$, there is now a whole continuous spectrum of prices $S_{t_k} \in \mathbb{R}$ possible. So it is not clear at all whether in this setting payoff replication is still possible. The answer is that in the limit $\Delta t \to 0$ exact payoff replication is still possible. To prove this, we need a slight generalization of the Ito-formula. For simplicity, we start with zero rates, $r = 0$.

In the first chapter we saw that the portfolio value V_{t_k} of a selffinancing strategy, which holds $\delta_{t_{k-1}}$ stocks 'at the end of time t_{k-1}' or 'at the beginning of time t_k' and readjusts this to δ_{t_k} stocks 'at the end of time t_k after the asset price has switched from $S_{t_{k-1}}$ to S_{t_k}, is given by

$$V_{t_k} \;=\; V_0 + \sum_{j=1}^{k} \delta_{t_{j-1}} \cdot (S_{t_j} - S_{t_{j-1}}) \;=\; V_{t_{k-1}} + \delta_{t_{k-1}} \cdot (S_{t_k} - S_{t_{k-1}}) \tag{3}$$

In continuous time with 'continuous trading' this may be rewritten as a stochastic integral, as an Ito-integral

$$V_t \;=\; V_0 \,+\, \int_0^t \delta_\tau\, dS_\tau \tag{4}$$

or in differential form, if we subtract the $V_{t_{k-1}}$-term on the right hand side (3),

$$dV \;=\; \delta\, dS \tag{5}$$

where dV is the limit of

$$V_t(S_t) - V_{t-\Delta t}(S_{t-\Delta t}) \;=\; V(S_t, t) - V(S_{t-\Delta t}, t - \Delta t) \;\overset{\Delta t \to 0}{\to}\; dV \tag{6}$$

Since we have $V = V(S_t, t)$ and S_t is a stochastic quantity, we have to use the Ito-Formula, the differential version of the Ito-Formula, to calculate the dV. Let's start by recalling the calculation rules for the Brownian motion,

$$\begin{aligned} (dx_t)^2 &= dt \\ dx_t\, dt &= 0 \\ (dt)^2 &= 0 \end{aligned} \tag{7}$$

From this, we derived already in week9b the following

Theorem 5.3.1 (Ito-Formula for Functions of a Brownian Motion): Let

$$F = F(x) \; : \; \mathbb{R} \to \mathbb{R}$$

be an arbitrary two-times differentiable function of one variable and let $\{x_t\}_{0 \leq t \leq T}$ be a Brownian motion. Then we have the following identities:

a) Differential Version: Let $dF(x_t) := F(x_t) - F(x_{t-dt})$. Then

$$\begin{aligned} dF(x_t) &= F'(x_t)\, dx_t + \tfrac{1}{2} F''(x_t)\, (dx_t)^2 \\ &= F'(x_t)\, dx_t + \tfrac{1}{2} F''(x_t)\, dt \end{aligned}$$

b) Integral Version: We have

$$F(x_T) - F(x_0) = \int_0^T F'(x_t)\, dx_t + \tfrac{1}{2} \int_0^T F''(x_t)\, dt$$

where the stochastic dx_t-integral above is to be defined as an Ito-integral according to

$$\int_0^T f(x_t)\, dx_t = \lim_{\Delta t \to 0} \sum_{k=1}^{N} f(x_{t_{k-1}})\, \Delta x_{t_k} = \lim_{\Delta t \to 0} \sum_{k=1}^{N} f(x_{t_{k-1}})\, \sqrt{\Delta t}\, \phi_k$$

and the Brownain motion $x_{t_{k-1}}$ at time $t_{k-1} = (k-1)\Delta t$ given by

$$x_{t_{k-1}} = \sqrt{\Delta t} \sum_{j=1}^{k-1} \phi_j \; .$$

A slightly generalized version of this is the following

Theorem 5.3.2 (Ito-Formula for Functions of a Brownian Motion and Time): Let

$$F = F(x, t) \; : \; \mathbb{R}^2 \to \mathbb{R}$$

be an arbitrary two-times differentiable function of two variables and let $\{x_t\}_{0 \leq t \leq T}$ be a Brownian motion. Then we have the following identities:

a) Differential Version: Let $dF(x_t,t) := F(x_t,t) - F(x_{t-dt}, t-dt)$. Then

$$\begin{aligned} dF &= \tfrac{\partial F}{\partial x}\, dx_t \;+\; \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}\,(dx_t)^2 \;+\; \tfrac{\partial F}{\partial t}\, dt \\ &= \tfrac{\partial F}{\partial x}\, dx_t \;+\; \Big\{ \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t} \Big\}\, dt \end{aligned}$$

b) Integral Version: We have

$$F(x_T,T) \;-\; F(x_0,0) \;=\; \int_0^T \tfrac{\partial F}{\partial x}\, dx_t \;+\; \int_0^T \Big\{ \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2} \;+\; \tfrac{\partial F}{\partial t} \Big\}\, dt$$

where the stochastic dx_t-integral above again is to be defined as an Ito-integral.

Since we want to calculate quantities like

$$V_t(S_t) - V_{t-\Delta t}(S_{t-\Delta t}) \;=\; V(S_t,t) - V(S_{t-\Delta t}, t-\Delta t) \;\stackrel{\Delta t \to 0}{\to}\; dV$$

where S is given by

$$S \;=\; S(x_t,t) \;=\; S_0\, e^{\sigma x_t + (\mu - \sigma^2/2)t}$$

we need actually a more general version than the two theorems above. Namely, the F above in the theorems is now the V, the portfolio value. However, we want to consider the V as a function of S_t, not of x_t. That is, we plug in stochastic objects, but not directly the Brownian motion, but functions of it. To specify the class of stochastic objects we can plug into the V or some $F = F(S_t,t)$, we need the following

Definition 5.3.1: An Ito diffusion is a stochastic process X_t given by the SDE

$$dX_t \;=\; a(X_t,t)\, dt \;+\; b(X_t,t)\, dx_t$$

with x_t being a Brownian motion.

Example: The Black-Scholes model given by the geometric Brownain motion

$$S_t \;=\; S(x_t,t) \;=\; S_0\, e^{\sigma x_t + (\mu - \sigma^2/2)t}$$

is an Ito-diffusion since with Theorem 5.3.2

$$\begin{aligned} dS_t &= \tfrac{\partial S}{\partial x}\, dx_t \;+\; \Big\{ \tfrac{1}{2}\tfrac{\partial^2 S}{\partial x^2} \;+\; \tfrac{\partial S}{\partial t} \Big\}\, dt \\ &= \sigma\, S_t\, dx_t \;+\; \Big\{ \tfrac{\sigma^2}{2}\, S_t \;+\; (\mu - \sigma^2/2)\, S_t \Big\}\, dt \\ &= \sigma\, S_t\, dx_t \;+\; \mu\, S_t\, dt \end{aligned}$$

which is equivalent to the Black-Scholes SDE

$$dS_t/S_t \;=\; \mu\, dt + \sigma\, dx_t$$

Thus we have

$$\begin{aligned} a(S_t, t) &= \mu\, S_t \\ b(S_t, t) &= \sigma\, S_t \end{aligned}$$

in Definition 5.3.1 and S_t is an Ito-diffusion. Now we can state a third theorem which summarizes the formulae we will actually use:

Theorem 5.3.3 (Ito-Formula for Functions of an Ito-Diffusion and Time): Let

$$F = F(x,t) \ : \ \mathbb{R}^2 \to \mathbb{R}$$

be an arbitrary two-times differentiable function of two variables and let $\{x_t\}_{0\le t\le T}$ be a Brownian motion. Let X_t be an Ito-diffusion given by the SDE

$$dX_t = a(X_t,t)\,dt + b(X_t,t)\,dx_t$$

We plug X_t into the first argument of F and consider the function $F = F(X_t,t)$. Then we have the following identities:

a) Differential Version: Let $dF(X_t,t) := F(X_t,t) - F(X_{t-dt}, t-dt)$ with X_t being the Ito-diffusion from above. Then

$$\begin{aligned} dF &= \tfrac{\partial F}{\partial x}\, dX_t + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}\,(dX_t)^2 + \tfrac{\partial F}{\partial t}\, dt \\ &= \tfrac{\partial F}{\partial x}\,(a\,dt + b\,dx_t) + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}\,(a\,dt + b\,dx_t)^2 + \tfrac{\partial F}{\partial t}\, dt \\ &= \tfrac{\partial F}{\partial x}\,(a\,dt + b\,dx_t) + \tfrac{b^2}{2}\tfrac{\partial^2 F}{\partial x^2}\, dt + \tfrac{\partial F}{\partial t}\, dt \\ &= \left\{ a\,\tfrac{\partial F}{\partial x} + \tfrac{b^2}{2}\tfrac{\partial^2 F}{\partial x^2} + \tfrac{\partial F}{\partial t} \right\} dt + b\,\tfrac{\partial F}{\partial x}\, dx_t \end{aligned}$$

b) Integral Version: We have

$$F(X_T,T) - F(X_0,0) = \int_0^T \left\{ a\,\tfrac{\partial F}{\partial x} + \tfrac{b^2}{2}\tfrac{\partial^2 F}{\partial x^2} + \tfrac{\partial F}{\partial t} \right\} dt + \int_0^T b\,\tfrac{\partial F}{\partial x}\, dx_t$$

where the stochastic dx_t-integral above again is to be defined as an Ito-integral.

Now we are in a position to calculate dV, the change of the value of the replicating portfolio in contiuous time. With the Ito-Formula, we get

$$\begin{aligned} dV &= \tfrac{\partial V}{\partial S}\, dS + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\,(dS)^2 + \tfrac{\partial V}{\partial t}\, dt \\ &= \tfrac{\partial V}{\partial S}\, dS + \left\{ \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \tfrac{\partial V}{\partial t} \right\} dt \qquad (8) \end{aligned}$$

Thus, if this change should be given by trading δ stocks of the underlying, that is, if this should be equal to $\delta\, dS$,

$$dV \;=\; \tfrac{\partial V}{\partial S}\, dS \;+\; \Big\{ \tfrac{1}{2} \tfrac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \Big\}\, dt \;\stackrel{!}{=}\; \delta\, dS \tag{9}$$

we have to have the equations

$$\delta \;=\; \tfrac{\partial V}{\partial S} \tag{10}$$

and

$$\tfrac{1}{2} \tfrac{\partial^2 V}{\partial S^2} \sigma^2 S^2 \;+\; \tfrac{\partial V}{\partial t} \;=\; 0 \tag{11}$$

which is the Black-Scholes equation for zero interest rates. Thus, if (10) and (11) are fulfilled, we can use the integral version of Theorem 5.3.3 with $X_t = S_t$ and $F(X_t, t) = V(S_t, t)$ and

$$(dS_t)^2 \;=\; S_t^2\, (\mu\, dt \;+\; \sigma\, dx_t)^2 \;\stackrel{\text{Rechenregeln BB}}{=}\; S_t^2\, \sigma^2\, dt \tag{12}$$

to obtain

$$\begin{aligned} V(S_T, T) \;-\; V(S_0, 0) \;&=\; \textstyle\int_0^T \tfrac{\partial V}{\partial S}\, dS_t \;+\; \int_0^T \underbrace{\Big\{ \tfrac{1}{2} \tfrac{\partial^2 V}{\partial S^2} \sigma^2 S_t^2 \;+\; \tfrac{\partial V}{\partial t} \Big\}}_{=\,0}\, dt \\ &=\; \textstyle\int_0^T \delta(S_t, t)\, dS_t \end{aligned} \tag{13}$$

Thus, some payoff $H = H(S_T)$ can be exactly replicated in continuous time if we impose the final condition

$$V(S_T, T) \;=\; H(S_T) \tag{14}$$

in addition to (11).

When interest rates are present, a similar derivation can be done. Since this is an important calculation and an important result, in the continuous time Black-Scholes model exact payoff replication is still possible, we state this in a separate theorem which we will formulate and prove in the next lecture.