Hochschule RheinMain
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Lösungen Übungsblatt 8
Finanzmathematik I

Aufgabe 1: In terms of the ϕ-variables, we have

$$\begin{aligned} I_1(\Delta t) &= \sum_{k=1}^{N} x_{t_{k-1}} \, (x_{t_k} - x_{t_{k-1}}) \\ &= \sum_{k=1}^{N} x_{t_{k-1}}(\phi_1, ..., \phi_{k-1}) \, \sqrt{\Delta t} \, \phi_k \end{aligned}$$

and

$$\begin{aligned} I_2(\Delta t) &= \sum_{k=1}^{N} \tfrac{x_{t_{k-1}} + x_{t_k}}{2} \, (x_{t_k} - x_{t_{k-1}}) \\ &= \tfrac{1}{2} I_1(\Delta t) \; + \; \tfrac{1}{2} \sum_{k=1}^{N} x_{t_k}(\phi_1, ..., \phi_k) \, \sqrt{\Delta t} \, \phi_k \end{aligned}$$

Let us introduce the notation

$$\mathsf{E}_k[\,\cdot\,] \;=\; \int_{\mathbb{R}} \;\cdot\; e^{-\frac{\phi_k^2}{2}} \, \tfrac{d\phi_k}{\sqrt{2\pi}}$$

and

$$\mathsf{E}_{\{1,...,k\}}[\,\cdot\,] \;=\; \int_{\mathbb{R}^k} \;\cdot\; \prod_{j=1}^{k} e^{-\frac{\phi_j^2}{2}} \, \tfrac{d\phi_j}{\sqrt{2\pi}}$$

such that

$$\mathsf{E}_{\{1,...,k\}}[\,\cdot\,] \;=\; \mathsf{E}_{\{1,...,k-1\}}\big[\, \mathsf{E}_k[\,\cdot\,] \,\big]$$

Since

$$\begin{aligned} \mathsf{E}[I_1(\Delta t)] &= \mathsf{E}\Big[\sum_{k=1}^{N} x_{t_{k-1}}(\phi_1, ..., \phi_{k-1}) \, \sqrt{\Delta t} \, \phi_k \Big] \\ &= \sum_{k=1}^{N} \mathsf{E}\Big[x_{t_{k-1}}(\phi_1, ..., \phi_{k-1}) \, \sqrt{\Delta t} \, \phi_k \Big] \\ &= \sum_{k=1}^{N} \mathsf{E}_{\{1,...,k\}}\Big[x_{t_{k-1}}(\phi_1, ..., \phi_{k-1}) \, \sqrt{\Delta t} \, \phi_k \Big] \\ &= \sum_{k=1}^{N} \mathsf{E}_{\{1,...,k-1\}}\big[x_{t_{k-1}}(\phi_1, ..., \phi_{k-1}) \big] \, \sqrt{\Delta t} \, \underbrace{\mathsf{E}_k[\phi_k]}_{=0} \;=\; 0 \end{aligned}$$

part (a) follows. To obtain part (b), observe that

$$x_{t_k} \;=\; x_{t_{k-1}} + \sqrt{\Delta t}\,\phi_k$$

and therefore

$$\begin{aligned}
&\mathsf{E}_{\{1,\ldots,k\}}\Big[x_{t_k}(\phi_1,\ldots,\phi_k)\,\sqrt{\Delta t}\,\phi_k\Big] \;=\\
&\quad=\;\mathsf{E}_{\{1,\ldots,k\}}\Big[\Big\{x_{t_{k-1}}(\phi_1,\ldots,\phi_{k-1}) \;+\; \sqrt{\Delta t}\,\phi_k\Big\}\,\sqrt{\Delta t}\,\phi_k\Big]\\
&\quad=\;\mathsf{E}_{\{1,\ldots,k\}}\Big[x_{t_{k-1}}(\phi_1,\ldots,\phi_{k-1})\,\sqrt{\Delta t}\,\phi_k\Big] \;+\; \Delta t\,\mathsf{E}_{\{1,\ldots,k\}}[\phi_k^2]\\
&\quad=\;0 \;+\; \Delta t\,\mathsf{E}_k[\phi_k^2] \;=\; \Delta t \times 1
\end{aligned}$$

which gives

$$\begin{aligned}
\mathsf{E}[I_2(\Delta t)] \;&=\; \mathsf{E}\Bigg[\tfrac{1}{2}I_1(\Delta t) \;+\; \tfrac{1}{2}\sum_{k=1}^{N} x_{t_k}(\phi_1,\ldots,\phi_k)\,\sqrt{\Delta t}\,\phi_k\Bigg]\\
&=\; \tfrac{1}{2}\sum_{k=1}^{N}\Delta t \;=\; \frac{N\Delta t}{2} \;=\; \frac{T}{2}\;.
\end{aligned}$$