Lösungen Übungsblatt 7
Finanzmathematik I

Aufgabe 1: We use Theorem 4.1 from the lecture notes which was the following statement:

Theorem 4.1: *Let $F : \mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$. Then*

$$\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) = \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \tag{1}$$

where the p-functions on the right hand side of (1) are given by

$$p_\tau(x, y) \;:=\; \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \;.$$

Since all expectations to be calculated depend only on the Brownian motion observed at a single time t, we have $m = 1$ for all integrals,

$$\begin{aligned} \mathsf{E}\big[F(x_t)\big] &= \int_{\mathbb{R}} F(x_t)\, p_t(0, x_t)\, dx_t \\ &= \int_{\mathbb{R}} F(x_t)\, \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x_t^2}{2t}}\, dx_t \\ &= \int_{\mathbb{R}} F(x)\, e^{-\frac{x^2}{2t}}\, \tfrac{dx}{\sqrt{2\pi t}} \end{aligned}$$

where in the last line we simply renamed the integration variable from x_t to x. If we substitute $v := x/\sqrt{t}$, $dv = dx/\sqrt{t}$, we may rewrite this as

$$\mathsf{E}\big[F(x_t)\big] \;=\; \int_{\mathbb{R}} F(\,\sqrt{t}\, v\,)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;. \tag{2}$$

Thus, with the formulae from Übungsblatt 5, we obtain:

a)

$$\begin{aligned} \mathsf{E}[\,x_t\,] &\overset{\substack{\text{Definiton} \\ \text{of } \mathsf{E}[\cdot]}}{=} \int x_t\, dW(\{x_t\}_{0<t\leq T}) \\ &\overset{(2)}{=} \int_{\mathbb{R}} \sqrt{t}\, v\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;=\; 0 \end{aligned}$$

$$
\begin{aligned}
\mathsf{E}[\,x_t^2\,] \;&\overset{\substack{\text{Definiton}\\ \text{of } \mathsf{E}[\,\cdot\,]}}{=}\; \int x_t^2 \; dW(\{x_t\}_{0<t\le T}) \\
&\overset{(2)}{=}\; \int_{\mathbb{R}} \big(\sqrt{t}\,v\big)^2 \, e^{-\frac{v^2}{2}} \, \frac{dv}{\sqrt{2\pi}} \\
&=\; t \int_{\mathbb{R}} v^2 \, e^{-\frac{v^2}{2}} \, \frac{dv}{\sqrt{2\pi}} \;=\; t
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{E}[\,x_t^n\,] \;&\overset{\substack{\text{Definiton}\\ \text{of } \mathsf{E}[\,\cdot\,]}}{=}\; \int x_t^n \; dW(\{x_t\}_{0<t\le T}) \\
&\overset{(2)}{=}\; \int_{\mathbb{R}} \big(\sqrt{t}\,v\big)^n \, e^{-\frac{v^2}{2}} \, \frac{dv}{\sqrt{2\pi}} \\
&=\; t^{\frac{n}{2}} \int_{\mathbb{R}} v^n \, e^{-\frac{v^2}{2}} \, \frac{dv}{\sqrt{2\pi}} \\
&=\; \begin{cases} (n-1)!! \times t^{\frac{n}{2}} & \text{falls } n \text{ gerade} \\ 0 & \text{falls } n \text{ ungerade} \end{cases}
\end{aligned}
$$

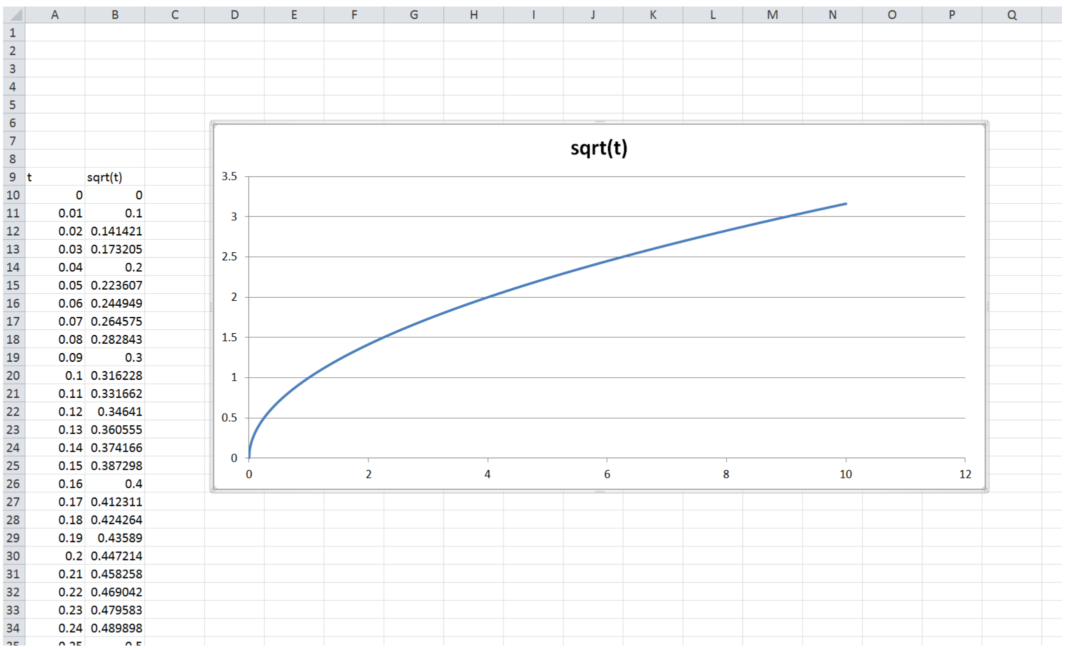
b) For the variance we obtain

$$
\begin{aligned}
\mathsf{V}[\,x_t\,] \;&=\; \mathsf{E}[\,x_t^2\,] \;-\; \big(\mathsf{E}[\,x_t\,]\big)^2 \\
&=\; t - 0 \;=\; t\,.
\end{aligned}
$$

such that the standard deviation is given by

$$
\sqrt{\mathsf{V}[\,x_t\,]} \;=\; \sqrt{t}
$$

which looks as follows:



Aufgabe 2: We use again formula (2) from above,

$$\mathsf{E}\big[F(x_t)\big] \;=\; \int_{\mathbb{R}} F(\,\sqrt{t}\,v\,)\,e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{2\pi}}\;,$$

to obtain

$$\begin{aligned}
\mathsf{E}[S_t] \;&=\; \mathsf{E}\Big[\,S_0\,e^{\mu t+\sigma x_t-\frac{\sigma^2}{2}t}\,\Big] \\
&\overset{(2)}{=}\; \int_{\mathbb{R}} S_0\,e^{\mu t+\sigma\sqrt{t}\,v-\frac{\sigma^2}{2}t}\,e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{2\pi}} \\
&=\; S_0\,e^{\mu t-\frac{\sigma^2}{2}t}\int_{\mathbb{R}} e^{\sigma\sqrt{t}\,v}\,e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{2\pi}} \\
&\overset{\text{UeBlatt 5}}{=}\; S_0\,e^{\mu t-\frac{\sigma^2}{2}t}\,e^{\frac{\sigma^2}{2}t} \;=\; S_0\,e^{\mu t}
\end{aligned}$$

And for the variance,

$$\begin{aligned}
\mathsf{V}[S_t] \;&=\; \mathsf{E}[S_t^2] \;-\; \big(\mathsf{E}[S_t]\big)^2 \\
&\overset{(2)}{=}\; \int_{\mathbb{R}} S_0^2\,e^{2\mu t+2\sigma\sqrt{t}\,v-\sigma^2 t}\,e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{2\pi}} \;-\; S_0^2\,e^{2\mu t} \\
&=\; S_0^2 e^{2\mu t-\sigma^2 t}\int_{\mathbb{R}} e^{2\sigma\sqrt{t}\,v}\,e^{-\frac{v^2}{2}}\,\frac{dv}{\sqrt{2\pi}} \;-\; S_0^2\,e^{2\mu t} \\
&\overset{\text{UeBlatt 5}}{=}\; S_0^2 e^{2\mu t-\sigma^2 t}\,e^{2\sigma^2 t} \;-\; S_0^2\,e^{2\mu t} \\
&=\; S_0^2\,e^{2\mu t}\big(\,e^{\sigma^2 t} \;-\; 1\,\big)\,.
\end{aligned}$$