

**Lösungen Übungsblatt 5
Finanzmathematik I**

Aufgabe 1: Gaussian Integrals:

a) $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} .$

Proof a) With polar coordinates

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

and

$$dx dy = r dr d\varphi$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx &= \left\{ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right\}^{1/2} \\ &= \left\{ \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx dy \right\}^{1/2} \\ &= \left\{ \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\varphi \right\}^{1/2} \\ &= \left\{ 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r dr \right\}^{1/2} \\ &= \left\{ 2\pi \left(e^{-\frac{r^2}{2}} \right) \Big|_0^{\infty} \right\}^{1/2} \\ &= \sqrt{2\pi} . \end{aligned}$$

b) $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} = 1 .$

Proof b) Follows directly from part (a) and the substitution of variables

$$y = \frac{x - \mu}{\sigma} \Rightarrow dy = \frac{dx}{\sigma}$$

since

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &\stackrel{(a)}{=} 1 . \end{aligned}$$

c-f) Offensichtlich sind die Formeln (c),(d) und (e) Spezialfälle von (f),

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \begin{cases} (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

mit $(n-1)!! = (n-1)(n-3)(n-5)\cdots 3 \cdot 1$.

Beweis f) With partial integration, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} &= \int_{-\infty}^{\infty} \underbrace{x^{n-1}}_{=f} \times \underbrace{x e^{-\frac{x^2}{2}}}_{=g'} \frac{dx}{\sqrt{2\pi}} \\ &= x^{n-1}(-e^{-\frac{x^2}{2}})|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (n-1)x^{n-2} (-e^{-\frac{x^2}{2}}) \frac{dx}{\sqrt{2\pi}} \\ &= 0 + (n-1) \int_{-\infty}^{\infty} x^{n-2} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

Thus the exponent n has decreased by 2. Repeating this procedure, we get

$$\begin{aligned} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} &= (n-1)(n-3)\cdots \begin{cases} 4 \cdot 2 \times \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} & \text{if } n \text{ is odd} \\ 3 \cdot 1 \times \int_{-\infty}^{\infty} 1 e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} & \text{if } n \text{ is even} \end{cases} \\ &= (n-1)(n-3)\cdots \begin{cases} 4 \cdot 2 \times 0 & \text{if } n \text{ is odd} \\ 3 \cdot 1 \times 1 & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)!! & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

g) $\int_{-\infty}^{\infty} e^{\lambda x} e^{-\alpha \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{\alpha}} e^{\frac{1}{\alpha} \frac{\lambda^2}{2}}, \quad \lambda \in \mathbb{R}, \alpha > 0.$

Proof g) By multiplying the above equation with $\sqrt{\alpha} e^{-\frac{1}{\alpha} \frac{\lambda^2}{2}}$ we have to show that

$$\sqrt{\alpha} e^{-\frac{1}{\alpha} \frac{\lambda^2}{2}} \int_{-\infty}^{\infty} e^{\lambda x} e^{-\alpha \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = 1.$$

This follows again from part (a), since the left hand side gives

$$\sqrt{\alpha} \int_{-\infty}^{\infty} e^{-\frac{1}{\alpha} \frac{\lambda^2}{2}} e^{\lambda x} e^{-\alpha \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \sqrt{\alpha} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}(\lambda^2 - 2\alpha\lambda x + \alpha^2 x^2)} \frac{dx}{\sqrt{2\pi}}$$

$$\begin{aligned} &= \sqrt{\alpha} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}(\lambda-\alpha x)^2} \frac{dx}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}y^2} \frac{dy}{\sqrt{2\pi}} \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} \frac{dv}{\sqrt{2\pi}} = 1. \end{aligned}$$