Lösungen 4. Übungsblatt
Finanzmathematik I

Aufgabe 1: **a)** We have

$$S_N \;=\; S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)$$

and, since all the returns are independent,

$$\begin{aligned} \mathsf{E}[\,S_N\,] \;&=\; \mathsf{E}\Big[S_0 \prod_{k=1}^{N} \big(1+\mathrm{ret}_k\big)\Big] \\ &=\; S_0 \prod_{k=1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_k]\big)\;. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathsf{E}[\mathrm{ret}_k] \;&=\; +q\times p_{\mathrm{up}} \;+\; (-q)\times p_{\mathrm{down}} \\ &=\; q/2 - q/2 \;=\; 0 \end{aligned}$$

Thus,

$$\mathsf{E}[\,S_N\,] \;=\; S_0 \prod_{k=1}^{N} \big(1+0\big) \;=\; S_0\;.$$

b) The notation $\mathsf{E}\big[\cdots \,\big|\, \{S_j\}_{j=0}^{k}\,\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned} S_N \;&=\; S_0 \prod_{j=1}^{N} \big(1+\mathrm{ret}_j\big) \\ &=\; S_0 \prod_{j=1}^{k} \big(1+\mathrm{ret}_j\big) \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \\ &=\; S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \end{aligned}$$

and obtain

$$\begin{aligned} \mathsf{E}\big[\,S_N \,\big|\, \{S_j\}_{j=0}^{k}\,\big] \;&=\; \mathsf{E}\Big[\,S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \,\big|\, \{S_j\}_{j=0}^{k}\,\Big] \\ &=\; S_k \prod_{j=k+1}^{N} \big(1+\mathsf{E}[\mathrm{ret}_j]\big) \\ &=\; S_k \prod_{j=k+1}^{N} \big(1+0\big) \;=\; S_k\;. \end{aligned}$$

c) Becaue of part (a), we have

$$\begin{aligned} \mathsf{E}\big[\,\frac{1}{N}\sum_{m=1}^{N} S_m\,\big] &= \frac{1}{N}\sum_{m=1}^{N}\mathsf{E}\big[\,S_m\,\big] \\ &= \frac{1}{N}\sum_{m=1}^{N} S_0 \;=\; S_0\ . \end{aligned}$$

d) Becaue of part (b), we obtain

$$\begin{aligned} \mathsf{E}\Big[\,\frac{1}{N}\sum_{m=1}^{N} S_m \,\big|\, \{S_j\}_{j=0}^{k}\Big] &= \frac{1}{N}\sum_{m=1}^{N}\mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^{k}\,\big] \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k}\mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^{k}\,\big] \;+\; \sum_{m=k+1}^{N}\mathsf{E}\big[\,S_m \,\big|\, \{S_j\}_{j=0}^{k}\,\big]\Big\} \\ &= \frac{1}{N}\Big\{\sum_{m=1}^{k} S_m \;+\; \sum_{m=k+1}^{N} S_k\Big\} \\ &= \frac{k}{N}\times\frac{1}{k}\sum_{m=1}^{k} S_m \;+\; \frac{N-k}{N}\times S_k\ . \end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N \;=\; S_0\prod_{k=1}^{N}\big(1+\mathrm{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} \;=\; \prod_{k=1}^{N}\frac{1}{1+\mathrm{ret}_k}\ .$$

Since all the returns are independent,

$$\begin{aligned} \mathsf{E}[\,S_0/S_N\,] &= \mathsf{E}\Big[\prod_{k=1}^{N}\frac{1}{1+\mathrm{ret}_k}\Big] \\ &= \prod_{k=1}^{N}\mathsf{E}\Big[\frac{1}{1+\mathrm{ret}_k}\Big] \\ &= \prod_{k=1}^{N}\Big\{\frac{1}{1+q}\times\frac{1}{2}+\frac{1}{1-q}\times\frac{1}{2}\Big\} \\ &= \prod_{k=1}^{N}\Big\{\frac{1}{1-q^2}\Big\} \\ &= \frac{1}{(1-q^2)^N}\ . \end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots\big|\,\{S_j\}_{j=0}^{k}\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write as in part (b)

$$S_0 \;/\; S_N \;=\; S_0 \;/\; \Big\{ S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \;\big|\; \{S_j\}_{j=0}^{k} \,\big] &= S_0/S_k\ \mathsf{E}\left[\prod_{m=k+1}^{N} \frac{1}{1+\mathrm{ret}_m} \;\Big|\; \{S_j\}_{j=0}^{k} \right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\left[\frac{1}{1+\mathrm{ret}_m} \right] \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{ \frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2} \right\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \left\{ \frac{1}{1-q^2} \right\} \\
&= S_0/S_k\ \frac{1}{(1-q^2)^{N-k}} \;.
\end{aligned}$$