Hochschule RheinMain WS 2023/24
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Lösungen 11. Übungsblatt
Finanzmathematik I

Aufgabe 1: Let us calculate more generally the price at some time $t \in [0,T]$. Then we have to use the following formula:

$$\begin{aligned}
V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H\big(S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\sigma^2/2)(T-t)}\big)\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}} \frac{S_0}{S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\sigma^2/2)(T-t)}}\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)}\,\tfrac{S_0}{S_t} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x - (r-\sigma^2/2)(T-t)}\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)}\,\tfrac{S_0}{S_t} e^{-(r-\sigma^2/2)(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-2r(T-t)}\,\tfrac{S_0}{S_t} e^{\sigma^2/2(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}}\, e^{-\frac{\sigma^2(T-t)}{2}} \tfrac{dx}{\sqrt{2\pi}}\; e^{\frac{\sigma^2(T-t)}{2}} \\
&= e^{(-2r+\sigma^2)(T-t)}\,\tfrac{S_0}{S_t} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^2} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{(-2r+\sigma^2)(T-t)}\;\tfrac{S_0}{S_t}\;\times 1 \\
&= \tfrac{S_0}{S_t}\, e^{(\sigma^2-2r)(T-t)}
\end{aligned}$$

At time $t=0$, this reduces to

$$V_0 \;=\; \tfrac{S_0}{S_0}\, e^{(\sigma^2-2r)(T-0)} \;=\; e^{(\sigma^2-2r)T} \;.$$

Aufgabe 2: Die allgemeine Pricing-Formel lautet

$$\begin{aligned}
V_0 &= e^{-rT} \int_{\mathbb{R}} H(S_T)\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}}
\end{aligned}$$

Wir müssen schauen, über welche x man tatsächlich integrieren muss:

$$\begin{aligned}
& S_T && \geq K \\
\Leftrightarrow\quad & S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T} && \geq K \\
\Leftrightarrow\quad & \sigma\sqrt{T}\,x + (r-\sigma^2/2)T && \geq \log[K/S_0] \\
\Leftrightarrow\quad & \sigma\sqrt{T}\,x && \geq \log[K/S_0] - (r-\sigma^2/2)T \\
\Leftrightarrow\quad & x && \geq \frac{\log[K/S_0] - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\
\Leftrightarrow\quad & x && \geq -\frac{\log[S_0/K] + (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\
\Leftrightarrow\quad & x && \geq -d
\end{aligned}$$

Also:

$$\begin{aligned}
V_0 &= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{-d}^{\infty} 1 \cdot e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{-\infty}^{+d} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT}\, N(d) \quad .
\end{aligned}$$