Hochschule RheinMain
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Lösungen 10. Übungsblatt
Finanzmathematik I

1. Aufgabe: **a)** The Black-Scholes equation for a european option with payoff $H(S_T)$ reads

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{1}$$

with the final condition

$$V(S,T) = H(S) \tag{2}$$

To turn (1) into a constant coefficient equation, we start by introducing the variables x and τ according to

$$\begin{aligned} x &= \frac{1}{\sigma} \log S \\ \tau &= T - t \end{aligned} \tag{3}$$

or

$$\begin{aligned} S &= e^{\sigma x} \\ t &= T - \tau \end{aligned} \tag{4}$$

and write

$$V(S,t) = V(e^{\sigma x}, T-\tau) =: e^{-r\tau} v(x,\tau) \tag{5}$$

Because of

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial \tau} \tag{6}$$

$$\frac{\partial}{\partial S} = \frac{1}{\sigma S}\frac{\partial}{\partial x} \tag{7}$$

$$\frac{\partial^2}{\partial S^2} = -\frac{1}{\sigma S^2}\frac{\partial}{\partial x} + \frac{1}{\sigma^2 S^2}\frac{\partial^2}{\partial x^2} \tag{8}$$

(1) becomes

$$rv - \frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2} S^2 \left(-\frac{1}{\sigma S^2}\frac{\partial v}{\partial x} + \frac{1}{\sigma^2 S^2}\frac{\partial^2 v}{\partial x^2} \right) + rS\,\frac{1}{\sigma S}\frac{\partial v}{\partial x} - rv = 0$$

or

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \tau} \tag{9}$$

b) Equation (9) looks almost like the diffusion equation. To eliminate the first derivative with respect to x we put

$$k \;:=\; \frac{r}{\sigma} - \frac{\sigma}{2} \tag{10}$$

and make the ansatz

$$v(x,\tau) \;=\; e^{-\alpha x - \beta\tau}\, u(x,\tau) \tag{11}$$

which gives

$$\frac{1}{2}\Big(\alpha^2 u \;-\; 2\alpha\frac{\partial u}{\partial x} \;+\; \frac{\partial^2 u}{\partial x^2}\Big) \;+\; k\Big(-\alpha u \;+\; \frac{\partial u}{\partial x}\Big) \;=\; -\beta u \;+\; \frac{\partial u}{\partial \tau}$$

$$\Leftrightarrow \qquad \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \;+\; (-\alpha + k)\,\frac{\partial u}{\partial x} \;+\; \Big(\frac{\alpha^2}{2} \;-\; \alpha k \;+\; \beta\Big) u \;=\; \frac{\partial u}{\partial \tau} \tag{12}$$

With the choice

$$\alpha \;=\; k \;=\; \frac{r}{\sigma} - \frac{\sigma}{2} \tag{13}$$

we get

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} \;+\; \Big(\frac{k^2}{2} \;-\; k^2 \;+\; \beta\Big) u \;=\; \frac{\partial u}{\partial \tau} \tag{14}$$

With the further choice of

$$\beta \;:=\; \frac{k^2}{2} \;=\; \frac{1}{2}\Big(\frac{r^2}{\sigma^2} \;-\; r \;+\; \frac{\sigma^2}{4}\Big) \tag{15}$$

we arrive at the diffusion equation for $u = u(x,\tau)$,

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2} \;=\; \frac{\partial u}{\partial \tau} \tag{16}$$

Since

$$\begin{aligned} u(x,\tau) \;&=\; e^{+\alpha x + \beta\tau}\, v(x,\tau) \\ &=\; e^{+\alpha x + \beta\tau}\, e^{+r\tau}\, V(\,e^{\sigma x}\,,\, T - \tau\,) \end{aligned} \tag{17}$$

and because of

$$\tau \;=\; 0 \quad \Leftrightarrow \quad t \;=\; T \tag{18}$$

we have the initial condition

$$\begin{aligned} u(x,0) \;&=\; e^{+\alpha x + \beta\cdot 0}\, e^{+r\cdot 0}\, V(\,e^{\sigma x}\,,\, T - 0\,) \\ &=\; e^{kx}\, H(\,e^{\sigma x}\,) \end{aligned} \tag{19}$$

c) We have

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial \tau} &= \left\{ \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right\} u \\
&= \left\{ \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right\} \int_{\mathbb{R}} u_0(y)\, \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}\, dy \\
&= \int_{\mathbb{R}} u_0(y) \left\{ \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right\} \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}\, dy
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{\partial}{\partial x}\tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} &= \tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{(-2)(x-y)}{2\tau}\, e^{-\frac{(x-y)^2}{2\tau}} = -\tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{x-y}{\tau}\, e^{-\frac{(x-y)^2}{2\tau}} \\
\frac{\partial^2}{\partial x^2}\tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} &= -\frac{\partial}{\partial x}\left\{ \tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{x-y}{\tau}\, e^{-\frac{(x-y)^2}{2\tau}} \right\} \\
&= -\tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{1}{\tau}\, e^{-\frac{(x-y)^2}{2\tau}} + \tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{(x-y)^2}{\tau^2}\, e^{-\frac{(x-y)^2}{2\tau}}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} &= -\tfrac{1}{2}\, \tfrac{1}{\sqrt{2\pi}\,\tau^{3/2}}\, e^{-\frac{(x-y)^2}{2\tau}} + \tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{(x-y)^2}{2\tau^2}\, e^{-\frac{(x-y)^2}{2\tau}} \\
&= \frac{1}{2}\left\{ -\tfrac{1}{\sqrt{2\pi\tau}\,\tau}\, e^{-\frac{(x-y)^2}{2\tau}} + \tfrac{1}{\sqrt{2\pi\tau}}\, \tfrac{(x-y)^2}{\tau^2}\, e^{-\frac{(x-y)^2}{2\tau}} \right\} \\
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\, \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}
\end{aligned}
$$

Thus,

$$
\left\{ \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial \tau} \right\} \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} = 0
$$

Furthermore, with the substitution of variables

$$
\begin{aligned}
v &= \tfrac{x-y}{\sqrt{\tau}} \\
y &= x - \sqrt{\tau}\, v \\
dy &= -\sqrt{\tau}\, dv
\end{aligned}
$$

we get

$$
\begin{aligned}
\textstyle\int_{-\infty}^{+\infty} u_0(y)\, \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}\, dy &= \textstyle\int_{+\infty}^{-\infty} u_0(x-\sqrt{\tau}\, v)\, \tfrac{-1}{\sqrt{2\pi}}\, e^{-\frac{v^2}{2}}\, dv \\
&= \textstyle\int_{-\infty}^{+\infty} u_0(x-\sqrt{\tau}\, v)\, \tfrac{+1}{\sqrt{2\pi}}\, e^{-\frac{v^2}{2}}\, dv \\
&\overset{\tau\to 0}{\to} \textstyle\int_{-\infty}^{+\infty} u_0(x)\, \tfrac{+1}{\sqrt{2\pi}}\, e^{-\frac{v^2}{2}}\, dv \\
&= u_0(x) \underbrace{\textstyle\int_{-\infty}^{+\infty} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{v^2}{2}}\, dv}_{=1} \\
&= u_0(x)
\end{aligned}
$$

Thus, also the correct initial condition is fulfilled.

2. Aufgabe: According to Aufgabe 1c we have

$$u(x,\tau) \;=\; \int_{\mathbb{R}} e^{ky}\, H(e^{\sigma y})\, \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}\, dy \tag{20}$$

Since

$$e^{-\alpha x-\beta\tau}\, e^{-r\tau}\, u(x,\tau) \;=\; V(\,e^{\sigma x}\,,\,T-\tau\,) \;=\; V(S,t)$$

we arrive at ($S = e^{\sigma x}$, $t = T-\tau$)

$$\begin{aligned}
V(S,t) \;&=\; e^{-kx-k^2\tau/2}\, e^{-r\tau} \;\times\; u(x,\tau) \\
&=\; e^{-kx-k^2\tau/2}\, e^{-r\tau} \;\times\; \int_{\mathbb{R}} e^{ky}\, H(e^{\sigma y})\, e^{-\frac{(x-y)^2}{2\tau}}\, \frac{dy}{\sqrt{2\pi\tau}} \\
&=\; e^{-(k^2/2+r)\tau} \;\times\; \int_{\mathbb{R}} e^{k(y-x)}\, H(e^{\sigma y})\, e^{-\frac{(x-y)^2}{2\tau}}\, \frac{dy}{\sqrt{2\pi\tau}} \\
&\overset{\substack{z\,=\,y-x\\ dz\,=\,dy}}{=}\; e^{-(k^2/2+r)\tau} \;\times\; \int_{\mathbb{R}} e^{kz}\, H(\,e^{\sigma x}\, e^{\sigma z}\,)\, e^{-\frac{z^2}{2\tau}}\, \frac{dz}{\sqrt{2\pi\tau}} \\
&=\; e^{-(k^2/2+r)\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma z}\,)\, e^{-\frac{z^2}{2\tau}+kz}\, \frac{dz}{\sqrt{2\pi\tau}} \\
&=\; e^{-(k^2/2+r)\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma\sqrt{\tau}\,z}\,)\, e^{-\frac{z^2}{2}+k\sqrt{\tau}\,z}\, \frac{dz}{\sqrt{2\pi}} \\
&=\; e^{-(k^2/2+r)\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma\sqrt{\tau}\,z}\,)\, e^{-\frac{(z-k\sqrt{\tau})^2}{2}+\frac{k^2\tau}{2}}\, \frac{dz}{\sqrt{2\pi}} \\
&=\; e^{-r\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma\sqrt{\tau}\,z}\,)\, e^{-\frac{(z-k\sqrt{\tau})^2}{2}}\, \frac{dz}{\sqrt{2\pi}}
\end{aligned}$$

Finally, we substitute

$$z \;-\; k\sqrt{\tau} \;\;=:\;\; y$$

to obtain

$$\begin{aligned}
V(S,t) \;&=\; e^{-r\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma\sqrt{\tau}\,(y+k\sqrt{\tau})}\,)\, e^{-\frac{y^2}{2}}\, \frac{dy}{\sqrt{2\pi}} \\
&=\; e^{-r\tau} \;\times\; \int_{\mathbb{R}} H(\,S\, e^{\sigma\sqrt{\tau}\,y+\sigma k\tau}\,)\, e^{-\frac{y^2}{2}}\, \frac{dy}{\sqrt{2\pi}}
\end{aligned}$$

Since

$$\sigma\, k \;=\; \sigma\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \;=\; r - \frac{\sigma^2}{2}$$

the statement follows.