

week6: Erwartungswerte mit Brownschen Bewegungen

Wir wollen uns jetzt überlegen, wie man allgemeine Erwartungswerte mit Brownschen Bewegungen konkret berechnen kann. Das Setting ist das folgende: Consider some discrete times t_k in the intervall $[0, T]$,

$$t_k = k \frac{T}{N} = k \Delta t, \quad k = 0, 1, \dots, N \quad (1)$$

where

$$N = N_T := \frac{T}{\Delta t} \in \mathbb{N} \quad (2)$$

and let

$$\phi_1, \phi_2, \dots, \phi_N \quad (3)$$

be independent normally distributed random numbers with mean zero and variance one,

$$\phi_k \in \mathcal{N}(0, 1) \quad \text{i.i.d.} \quad (4)$$

Then a Brownian motion at discretized time $t = t_k = k \Delta t$ is given by the combination of integration variables

$$x_t = x_{t_k} := \sqrt{\Delta t} \sum_{j=1}^k \phi_j \quad (5)$$

We start with the following expectation which involves only one Brownian motion observed at one single time $t = T = t_N$:

$$\begin{aligned} \mathbb{E}[f(x_T)] &= \mathbb{E}[f(x_{t_N})] = \mathbb{E}\left[f\left(\sqrt{\Delta t} \sum_{k=1}^N \phi_k\right)\right] \\ &= \int_{\mathbb{R}^N} f\left(\sqrt{\Delta t} \sum_{k=1}^N \phi_k\right) \prod_{k=1}^N \frac{1}{\sqrt{2\pi}} e^{-\frac{\phi_k^2}{2}} d\phi_k \end{aligned} \quad (6)$$

Here f is an arbitrary function. We make a substitution of variables $(\phi_k)_{1 \leq k \leq N} \rightarrow (x_k)_{1 \leq k \leq N}$ defined as follows:

$$\begin{aligned} x_1 &= \sqrt{\Delta t} \phi_1 & \phi_1 &= x_1 / \sqrt{\Delta t} \\ x_2 &= \sqrt{\Delta t} (\phi_1 + \phi_2) & \phi_2 &= (x_2 - x_1) / \sqrt{\Delta t} \\ x_3 &= \sqrt{\Delta t} (\phi_1 + \phi_2 + \phi_3) & \phi_3 &= (x_3 - x_2) / \sqrt{\Delta t} \\ &\vdots & &\vdots \\ x_N &= \sqrt{\Delta t} (\phi_1 + \phi_2 + \dots + \phi_N) & \phi_N &= (x_N - x_{N-1}) / \sqrt{\Delta t} \end{aligned} \quad (7)$$

The Jacobian of the transformation (7) is $\det \frac{\partial \phi}{\partial x} = 1/\sqrt{\Delta t}^N$ since

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \begin{pmatrix} - & \nabla_x \phi_1 & - \\ & \vdots & \\ - & \nabla_x \phi_N & - \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_N} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_N}{\partial x_1} & \frac{\partial \phi_N}{\partial x_2} & \cdots & \frac{\partial \phi_N}{\partial x_N} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 & 0 \\ -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ 0 & & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} & 0 \\ 0 & 0 & \cdots & -\frac{1}{\sqrt{\Delta t}} & \frac{1}{\sqrt{\Delta t}} \end{pmatrix} \end{aligned}$$

Thus the expectation (6) becomes, with $N = N_t = t/\Delta t$,

$$\begin{aligned} \mathbb{E} \left[f \left(\sqrt{\Delta t} \sum_{k=1}^N \phi_k \right) \right] &= \int_{\mathbb{R}^N} f \left(\sqrt{\Delta t} \sum_{k=1}^N \phi_k \right) \prod_{k=1}^N \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{\phi_k^2}{2}} d\phi_k \right\} \\ &= \int_{\mathbb{R}^N} f \left(\sqrt{\Delta t} \sum_{k=1}^N \phi_k \right) \prod_{k=1}^N \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{\phi_k^2}{2}} \right\} d\phi_1 \cdots d\phi_N \\ &= \int_{\mathbb{R}^N} f(x_N) \prod_{k=1}^N \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \right\} \det \left[\frac{\partial \phi}{\partial x} \right] dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(x_N) \prod_{k=1}^N \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \right\} \frac{1}{(\sqrt{\Delta t})^N} dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(x_N) \prod_{k=1}^N \left\{ \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(x_{k-1} - x_k)^2}{2\Delta t}} \right\} dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f(x_N) \prod_{k=1}^N \left\{ p_{\Delta t}(x_{k-1}, x_k) dx_k \right\} \end{aligned} \tag{8}$$

where we introduced the kernel

$$p_\tau(x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-y)^2}{2\tau}} \tag{9}$$

and used the definition

$$x_0 := 0 \tag{10}$$

The kernel (9) has the following basic property:

Lemma 6.1: *Let $p_t(x, y)$ be given by (9). Then*

$$\int_{\mathbb{R}} p_s(x, y) p_t(y, z) dy = p_{s+t}(x, z) \tag{11}$$

Proof: We have

$$\begin{aligned}
p_s(x, y)p_t(y, z) &= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}} e^{-\left(\frac{1}{2s} + \frac{1}{2t}\right)y^2 + \left(\frac{x}{s} + \frac{z}{t}\right)y} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}} e^{-\frac{s+t}{2st}(y^2 - 2\frac{xt+zs}{s+t}y)} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} e^{\frac{s+t}{2st}(\frac{xt+zs}{s+t})^2} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} e^{\frac{(xt+zs)^2}{2st(s+t)}} \\
&= \frac{1}{2\pi\sqrt{st}} e^{-x^2(\frac{1}{2s} - \frac{t}{2s(s+t)}) - z^2(\frac{1}{2t} - \frac{s}{2t(s+t)}) + \frac{xz}{s+t}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}} e^{-\frac{x^2}{2(s+t)} - \frac{z^2}{2(s+t)} + \frac{xz}{s+t}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}} e^{-\frac{(x-z)^2}{2(s+t)}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2}
\end{aligned}$$

Also

$$\begin{aligned}
\int_{\mathbb{R}} p_s(x, y)p_t(y, z) dy &= \frac{1}{2\pi\sqrt{st}} e^{-\frac{(x-z)^2}{2(s+t)}} \int_{\mathbb{R}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} dy \\
&= \frac{1}{2\pi\sqrt{st}} e^{-\frac{(x-z)^2}{2(s+t)}} \int_{\mathbb{R}} e^{-\frac{s+t}{2st}v^2} dv \\
&= \frac{1}{2\pi\sqrt{st}} e^{-\frac{(x-z)^2}{2(s+t)}} \sqrt{2\pi} \sqrt{\frac{st}{s+t}} \\
&= \frac{1}{\sqrt{2\pi(s+t)}} e^{-\frac{(x-z)^2}{2(s+t)}} \\
&= p_{s+t}(x, z)
\end{aligned}$$

which proves the lemma. ■

Using this lemma, we can actually perform the integrals over $x_1, x_2, \dots, x_{N-1}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \cdots \int_{\mathbb{R}} dx_{N-1} \underbrace{p_{\Delta t}(x_0, x_1) p_{\Delta t}(x_1, x_2) p_{\Delta t}(x_2, x_3) \cdots p_{\Delta t}(x_{N-1}, x_N)}_{\int dx_1 \rightarrow p_{2\Delta t}(x_0, x_2)} \\
&= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1} \underbrace{p_{2\Delta t}(x_0, x_2) p_{\Delta t}(x_2, x_3) \cdots p_{\Delta t}(x_{N-1}, x_N)}_{\int dx_2 \rightarrow p_{3\Delta t}(x_0, x_3)} \\
&= \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1} p_{3\Delta t}(x_0, x_3) \cdots p_{\Delta t}(x_{N-1}, x_N) \\
&= \int_{\mathbb{R}} dx_{N-1} p_{(N-1)\Delta t}(x_0, x_{N-1}) p_{\Delta t}(x_{N-1}, x_N) \\
&= p_{N\Delta t}(x_0, x_N)
\end{aligned}$$

Thus (8) simplifies to

$$\begin{aligned}
\mathbb{E} \left[f \left(\sqrt{\Delta t} \sum_{k=1}^{N_t} \phi_k \right) \right] &= \int_{\mathbb{R}^N} f(x_N) \prod_{k=1}^N \left\{ p_{\Delta t}(x_{k-1}, x_k) dx_k \right\} \\
&= \int_{\mathbb{R}} f(x_N) \times p_{N\Delta t}(x_0, x_N) dx_N \\
&\stackrel[N\Delta t = T]{x_0 = 0} = \int_{\mathbb{R}} f(x) \times \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx
\end{aligned} \tag{12}$$

Instead of labelling the x with $k \in \{1, 2, \dots, N\}$, we label them with $t_k := k\Delta t$ which has the meaning of time. In particular, $t_N = N\Delta t = T$. So, we rename $x_k \rightarrow x_{k\Delta t} = x_{t_k}$. With that, we write down the following very important

Definition 6.1: Let $N_T = T/\Delta t$ and $p_t(x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$. Then the measure

$$dW(\{x_t\}_{0 < t \leq T}) := \lim_{\Delta t \rightarrow 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) dx_{k\Delta t} \tag{13}$$

is called the Wiener measure and the family of random variables or integration variables $\{x_t\}_{0 < t \leq T}$ is called a Brownian motion. In terms of i.i.d. random variables $\phi_k \in \mathcal{N}(0, 1)$,

$$x_t := \lim_{\Delta t \rightarrow 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \phi_k \tag{14}$$

Remark: The time discretized version of the Wiener measure, this is what we actually will usually use, is simply given by a product of Gaussian normal distributions. With $N = N_T = T/\Delta t$ and $t_k = k\Delta t$,

$$\begin{aligned}
dW(\{x_{t_k}\}_{0 < k \leq N}) &= \prod_{k=1}^N p_{\Delta t}(x_{t_{k-1}}, x_{t_k}) dx_{t_k} \\
&= \prod_{k=1}^N \left\{ e^{-\frac{(x_{t_k} - x_{t_{k-1}})^2}{2\Delta t}} \frac{dx_{t_k}}{\sqrt{2\pi\Delta t}} \right\} \\
&= \prod_{k=1}^N \left\{ e^{-\frac{\phi_k^2}{2}} \frac{d\phi_k}{\sqrt{2\pi}} \right\}
\end{aligned} \tag{15}$$

where the last line in (15) is due to the substitution of variables, which is equivalent to the definition of Brownian motion in discrete time,

$$x_{t_k} = \sqrt{\Delta t} \sum_{j=1}^k \phi_j \tag{16}$$

from which we get the recursion

$$x_{t_k} = x_{t_{k-1}} + \sqrt{\Delta t} \phi_k \tag{17}$$

The formulae (15,16,17) are very important and will be used over and over again.

Integrals with respect to the Wiener measure are computed according to the following

Theorem 6.1: *Let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \dots < t_m \leq T$ be some times and $x_{t_1}, \dots, x_{t_m}$ be a Brownian motion observed at those times. Then*

$$\begin{aligned} \mathbb{E}[F(x_{t_1}, \dots, x_{t_m})] &\stackrel{\text{notation}}{=} \int F(x_{t_1}, \dots, x_{t_m}) dW(\{x_t\}_{0 \leq t \leq T}) \\ &\stackrel{\text{statement theorem}}{=} \int_{\mathbb{R}^m} F(x_{t_1}, \dots, x_{t_m}) \prod_{\ell=1}^m p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell}) dx_{t_\ell} \end{aligned} \quad (18)$$

Proof: The calculation is very similar to the calculation which lead to (12). Because of (11) only the $x_{t_1}, \dots, x_{t_m}$ integration variables survive. We are a bit less explicit and a bit more compact now and write

$$\begin{aligned} &\int F(x_{t_1}, \dots, x_{t_m}) dW(\{x_t\}_{0 \leq t \leq T}) \\ &= \lim_{\Delta t \rightarrow 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, \dots, x_{t_m}) \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) dx_{k\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, \dots, x_{t_m}) \prod_{k=1}^{N_{t_1}} p_{\Delta t}(\dots) dx_{k\Delta t} \prod_{k=N_{t_1}+1}^{N_{t_2}} p_{\Delta t}(\dots) dx_{k\Delta t} \times \dots \\ &\quad \dots \times \prod_{k=N_{t_{m-1}}+1}^{N_{t_m}} p_{\Delta t}(\dots) dx_{k\Delta t} \\ &\stackrel{(11)}{=} \lim_{\Delta t \rightarrow 0} \int_{\mathbb{R}^m} F(x_{t_1}, \dots, x_{t_m}) p_{t_1}(x_0, x_{t_1}) dx_{t_1} p_{t_2-t_1}(x_{t_1}, x_{t_2}) dx_{t_2} \times \dots \\ &\quad \dots \times p_{t_m-t_{m-1}}(x_{t_{m-1}}, x_{t_m}) dx_{t_m} \\ &= \int_{\mathbb{R}^m} F(x_{t_1}, \dots, x_{t_m}) \prod_{\ell=1}^m p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell}) dx_{t_\ell} \end{aligned}$$

which coincides with (18). ■