

Lösungen Übungsblatt 6 Finanzmathematik II

Aufgabe 1: We use Theorem 6.1 from week6.pdf which was the following statement:

Theorem 6.1: Let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \dots < t_m \leq T$. Then

$$\int F(x_{t_1}, \dots, x_{t_m}) dW(\{x_t\}_{0 \leq t \leq T}) = \int_{\mathbb{R}^m} F(x_{t_1}, \dots, x_{t_m}) \prod_{\ell=1}^m p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell}) dx_{t_\ell} \quad (1)$$

where the p -functions on the right hand side of (1) are given by

$$p_\tau(x, y) := \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-y)^2}{2\tau}}.$$

Since all expectations to be calculated depend only on the Brownian motion observed at a single time t , we have $m = 1$ for all integrals,

$$\begin{aligned} \mathbb{E}[F(x_t)] &= \int_{\mathbb{R}} F(x_t) p_t(0, x_t) dx_t \\ &= \int_{\mathbb{R}} F(x_t) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x_t^2}{2t}} dx_t \\ &= \int_{\mathbb{R}} F(x) e^{-\frac{x^2}{2t}} \frac{dx}{\sqrt{2\pi t}} \end{aligned}$$

where in the last line we simply renamed the integration variable from x_t to x . If we substitute $v := x/\sqrt{t}$, $dv = dx/\sqrt{t}$, we may rewrite this as

$$\mathbb{E}[F(x_t)] = \int_{\mathbb{R}} F(\sqrt{t} v) e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}}. \quad (2)$$

Thus, with the formulae from Übungsblatt 5, FMI, we obtain:

a)

$$\begin{aligned} \mathbb{E}[x_t] &\stackrel{\text{Definition of } \mathbb{E}[\cdot]}{=} \int x_t dW(\{x_t\}_{0 \leq t \leq T}) \\ &\stackrel{(2)}{=} \int_{\mathbb{R}} \sqrt{t} v e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} = 0 \end{aligned}$$

$$\begin{aligned}
\mathbb{E}[x_t^2] &\stackrel{\text{Definiton of } \mathbb{E}[\cdot]}{=} \int x_t^2 dW(\{x_t\}_{0 < t \leq T}) \\
&\stackrel{(2)}{=} \int_{\mathbb{R}} (\sqrt{t} v)^2 e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\
&= t \int_{\mathbb{R}} v^2 e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} = t \\
\\
\mathbb{E}[x_t^n] &\stackrel{\text{Definiton of } \mathbb{E}[\cdot]}{=} \int x_t^n dW(\{x_t\}_{0 < t \leq T}) \\
&\stackrel{(2)}{=} \int_{\mathbb{R}} (\sqrt{t} v)^n e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\
&= t^{\frac{n}{2}} \int_{\mathbb{R}} v^n e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\
&= \begin{cases} (n-1)!! \times t^{\frac{n}{2}} & \text{falls } n \text{ gerade} \\ 0 & \text{falls } n \text{ ungerade} \end{cases}
\end{aligned}$$

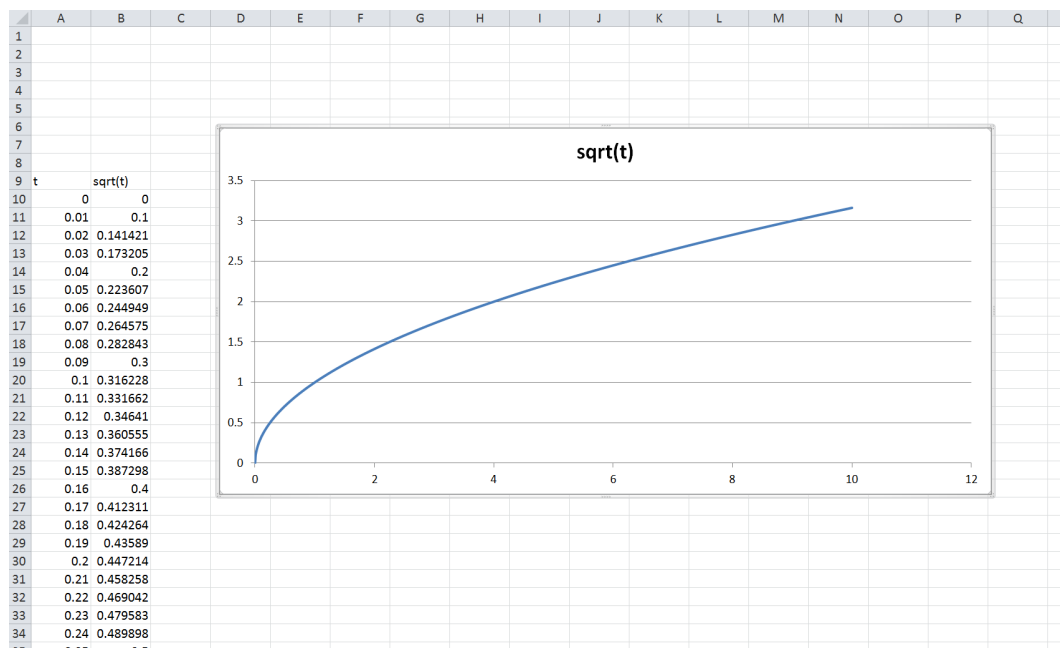
b) For the variance we obtain

$$\begin{aligned}
V[x_t] &= \mathbb{E}[x_t^2] - (\mathbb{E}[x_t])^2 \\
&= t - 0 = t.
\end{aligned}$$

such that the standard deviation is given by

$$\sqrt{V[x_t]} = \sqrt{t}$$

which looks as follows:



Aufgabe 2: We use again formula (2) from above,

$$\mathbb{E}[F(x_t)] = \int_{\mathbb{R}} F(\sqrt{t}v) e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}},$$

to obtain

$$\begin{aligned} \mathbb{E}[S_t] &= \mathbb{E}\left[S_0 e^{\mu t + \sigma x_t - \frac{\sigma^2}{2}t}\right] \\ &\stackrel{(2)}{=} \int_{\mathbb{R}} S_0 e^{\mu t + \sigma \sqrt{t}v - \frac{\sigma^2}{2}t} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\ &= S_0 e^{\mu t - \frac{\sigma^2}{2}t} \int_{\mathbb{R}} e^{\sigma \sqrt{t}v} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} \\ &\stackrel{\text{UeBlatt 5}}{=} S_0 e^{\mu t - \frac{\sigma^2}{2}t} e^{\frac{\sigma^2}{2}t} = S_0 e^{\mu t} \end{aligned}$$

And for the variance,

$$\begin{aligned} V[S_t] &= \mathbb{E}[S_t^2] - (\mathbb{E}[S_t])^2 \\ &\stackrel{(2)}{=} \int_{\mathbb{R}} S_0^2 e^{2\mu t + 2\sigma \sqrt{t}v - \sigma^2 t} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} - S_0^2 e^{2\mu t} \\ &= S_0^2 e^{2\mu t - \sigma^2 t} \int_{\mathbb{R}} e^{2\sigma \sqrt{t}v} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} - S_0^2 e^{2\mu t} \\ &\stackrel{\text{UeBlatt 5}}{=} S_0^2 e^{2\mu t - \sigma^2 t} e^{2\sigma^2 t} - S_0^2 e^{2\mu t} \\ &= S_0^2 e^{2\mu t} (e^{\sigma^2 t} - 1). \end{aligned}$$