The Dynamics of the Hubbard Model

through

Stochastic Calculus

and

Girsanov Transformation

Detlef Lehmann, Hochschule RheinMain, Faculty of Engineering
Postfach 3251, 65022 Wiesbaden, Germany

detlef.lehmann@hs-rm.de

https://www.hs-rm.de/de/hochschule/personen/lehmann-detlef/

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45/91 slides are pictures or title/cover slides or appendix, motivation has 21 pictures

1. Setup and Motivation

1. Setup and Motivation

Bose-Hubbard Hamiltonian in d Dimensions (n.n. = nearest neighbors):

$$H \;=\; -J \sum_{\substack{i,j \\ \text{n.n.}}} \left(a_i^+ a_j \;+\; a_j^+ a_i \right) \;+\; \frac{U}{2} \sum_j a_j^+ a_j^+ a_j \; a_j \;+\; \sum_j \epsilon_j \; a_j^+ a_j$$

Bosonic Annihilation and Creation operators:

$$[\, a_i \,, a_j^+] \;=\; \delta_{i,j}$$

Cubic Lattice Γ:

$$j \;=\; (j_1, \cdots, j_d) \;\in\; \{1, 2, \cdots, L\}^d \;=:\; \Gamma$$

1. Setup and Motivation: Quantities of Interest

Initial state: N particles at some lattice site j_0

$$\psi_0 \quad := \quad \frac{(a_{j_0}^+)^N}{\sqrt{N!}} \, |0\rangle$$

Time evolved state:

$$\psi_t \quad = \quad e^{-itH} \, \psi_0$$

Number of particles $\langle n_{j,t} \rangle$ at lattice site j at some time $t > 0$:

$$\langle n_{j,t} \rangle \quad := \quad \langle \psi_t, a_j^+ a_j \, \psi_t \rangle$$

Already for just two lattice sites 1 and 2, quite complex behaviour, Example:

1. Setup and Motivation: Example Two Site Bose-Hubbard Model

Two Site Bose-Hubbard model:

$$H \;=\; \varepsilon \left(a_1^+ a_2 \;+\; a_2^+ a_1 \right) \;+\; u \left(a_1^+ a_1^+ a_1 \, a_1 \;+\; a_2^+ a_2^+ a_2 \, a_2 \right)$$

$N = 20$ particles at lattice site $j_0 = 1$:

$$\psi_0 \;:=\; \frac{(a_1^+)^N}{\sqrt{N!}} \, |0\rangle$$

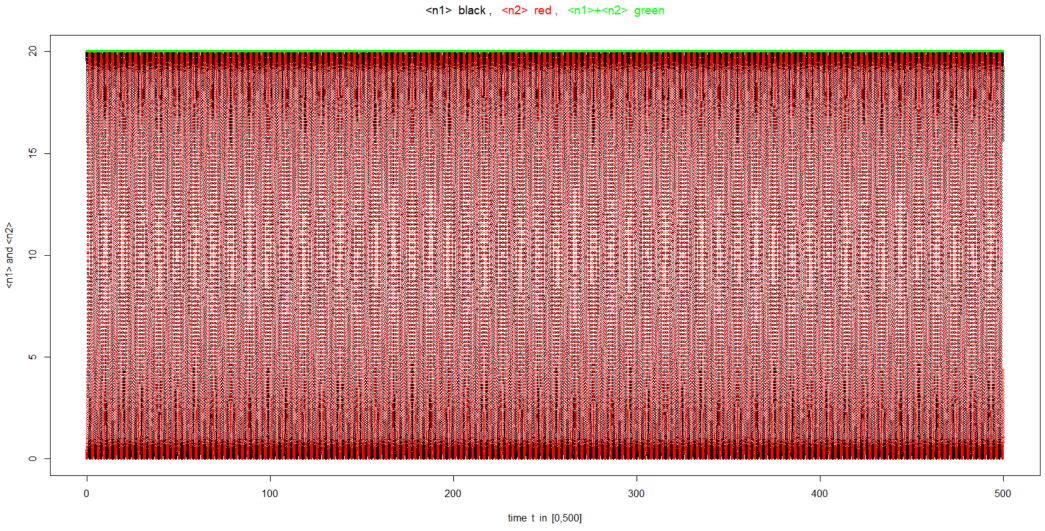
Plot $\langle n_{1,t} \rangle = \langle \psi_t, a_1^+ a_1 \, \psi_t \rangle$ for

$$\begin{aligned} \varepsilon &= 1 \\ t &\in [0, 500] \\ N &= 20 \\ u &= g/N \end{aligned}$$

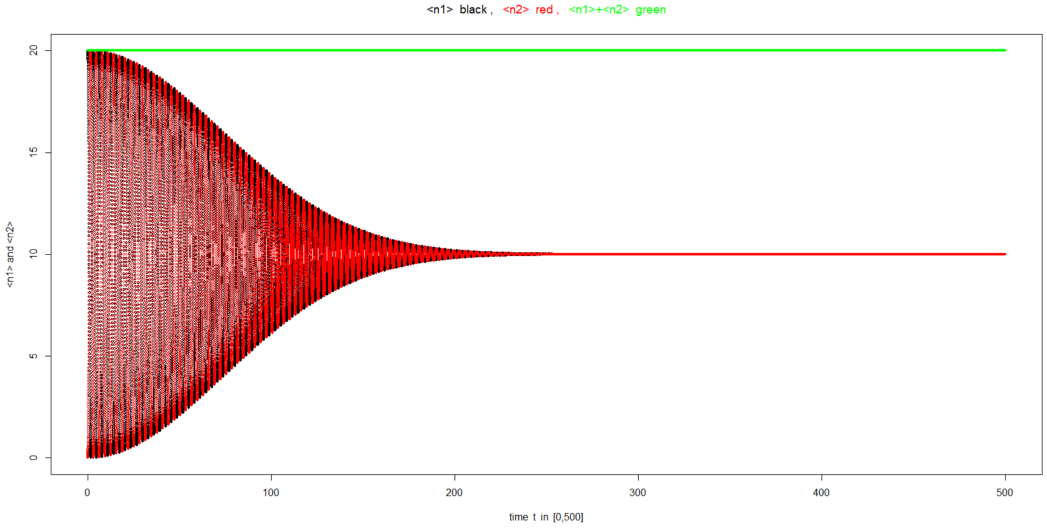
with interaction strength

$$g \;\in\; \left\{ 0 \,,\, \tfrac{1}{16} \,,\, \tfrac{1}{8} \,,\, \tfrac{1}{4} \,,\, \tfrac{1}{2} \,,\, 1 \,,\, 1.5 \,,\, 2.0 \,,\, 2.5 \,,\, 3 \,,\, 4 \,,\, 5 \,,\, 25 \right\}$$

$g = 0$, non interacting case, $\langle n_{1,t} \rangle = N \cos^2 \varepsilon t$



$$g = 1/16$$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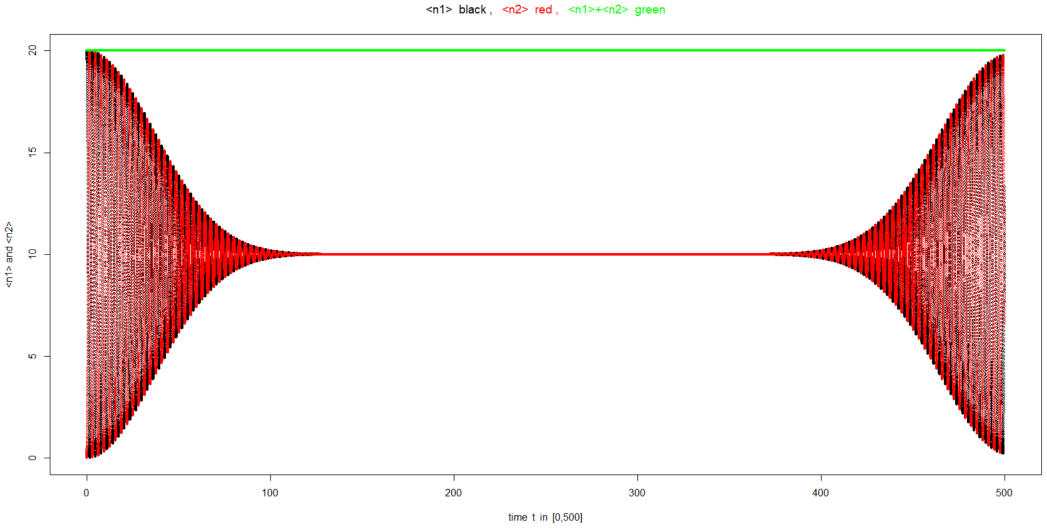
time t in [0,500]

$$g = 1/8$$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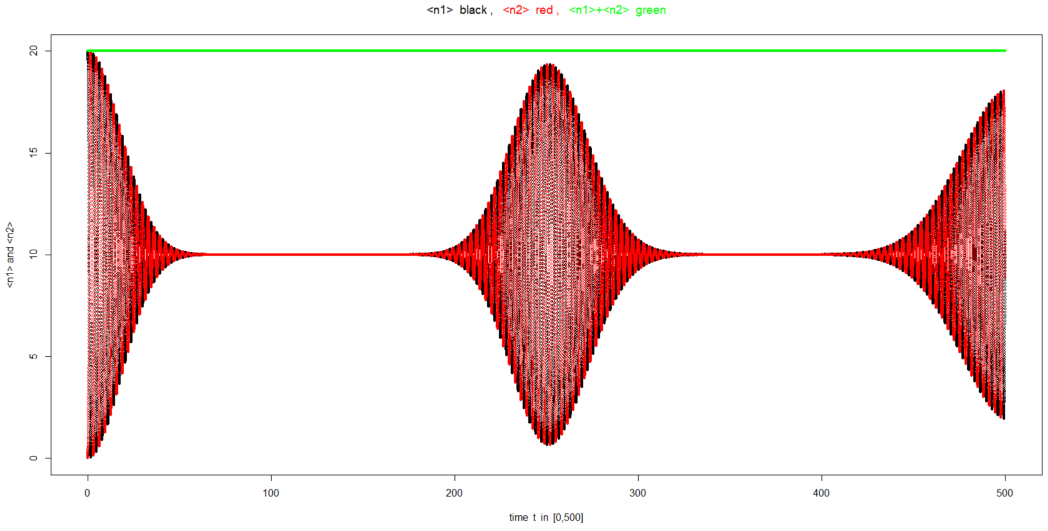
$$g = 1/4$$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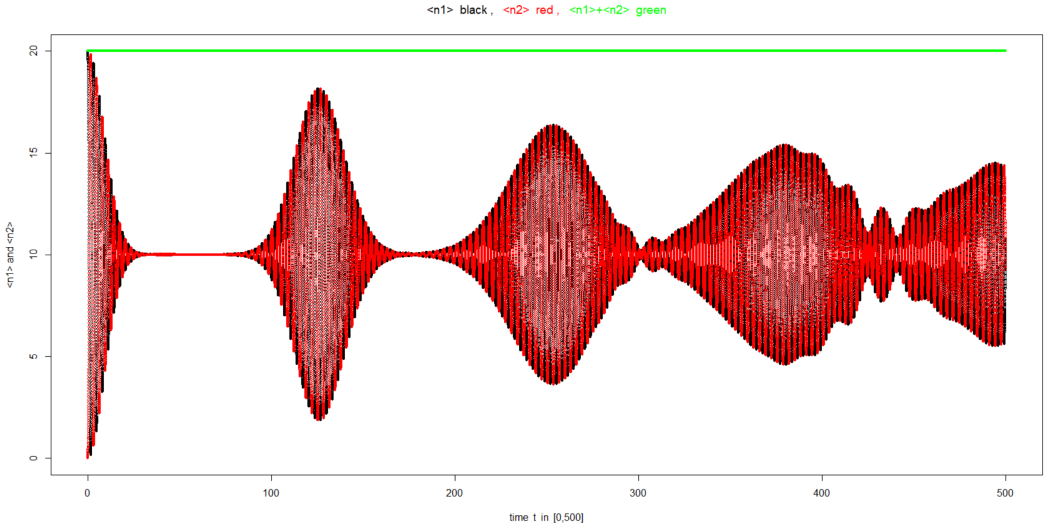
$$g = 1/2$$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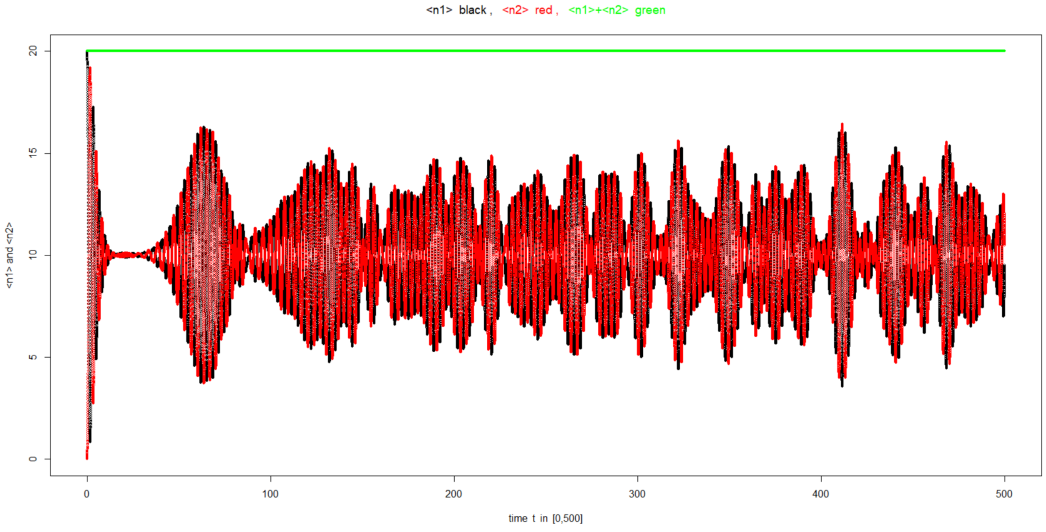
$$g = 1$$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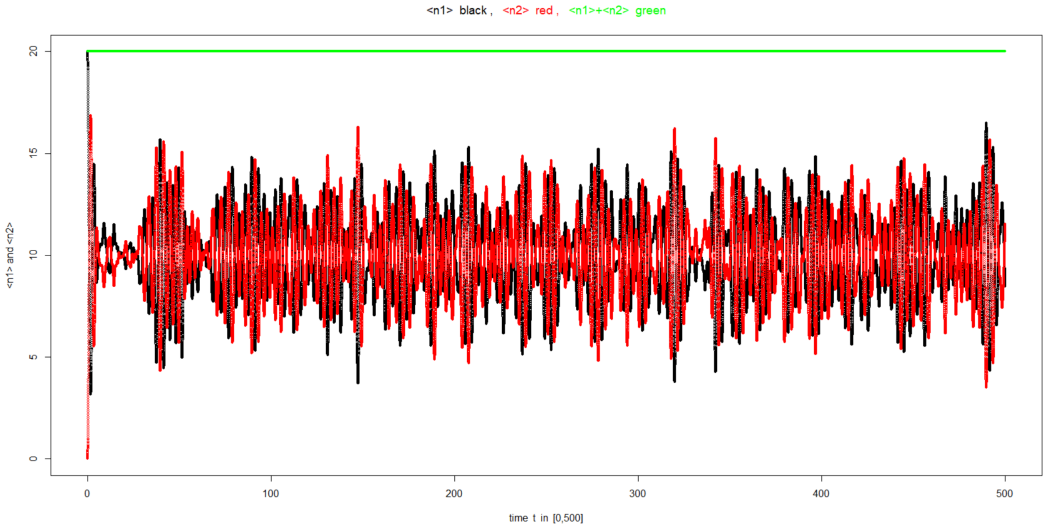
$$g = 1.5$$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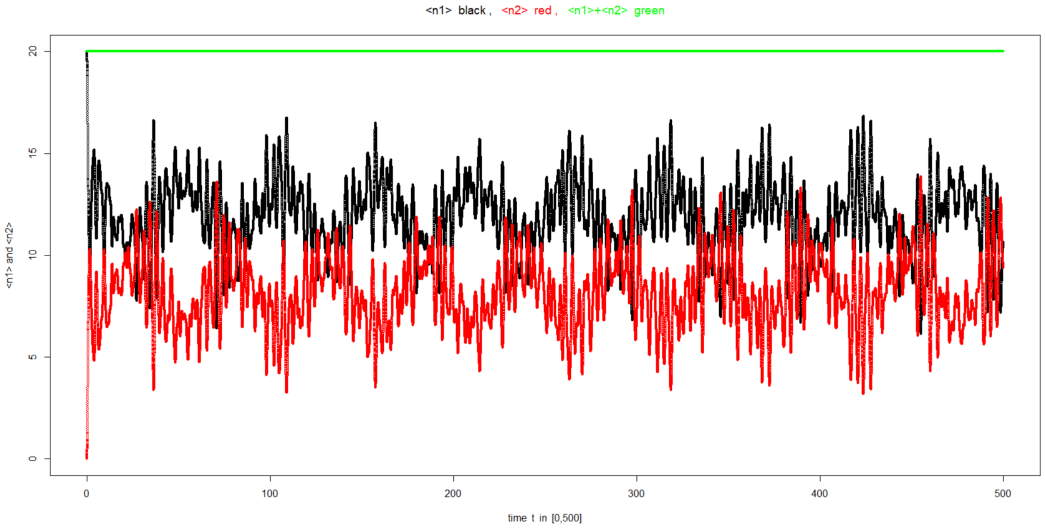
$g = 2.0$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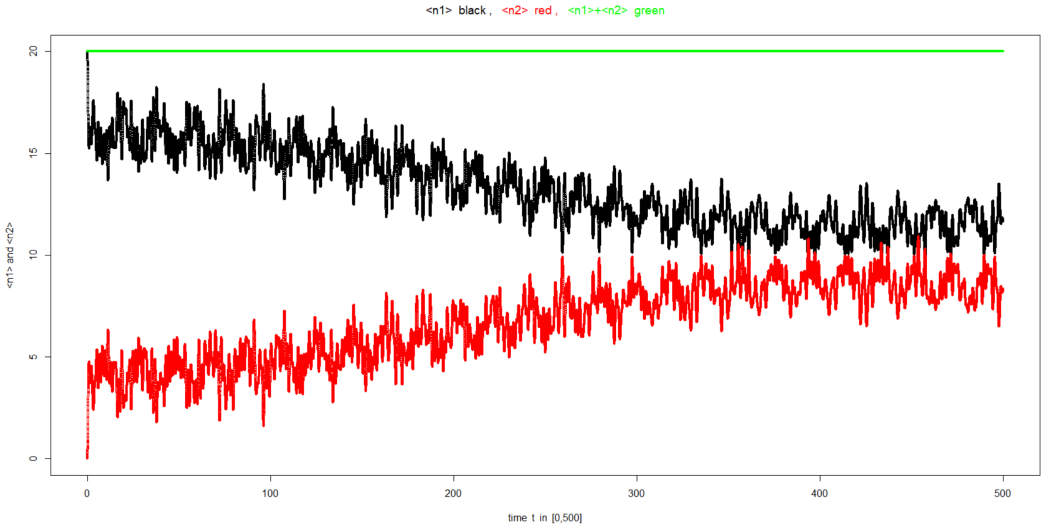
$$g = 2.5$$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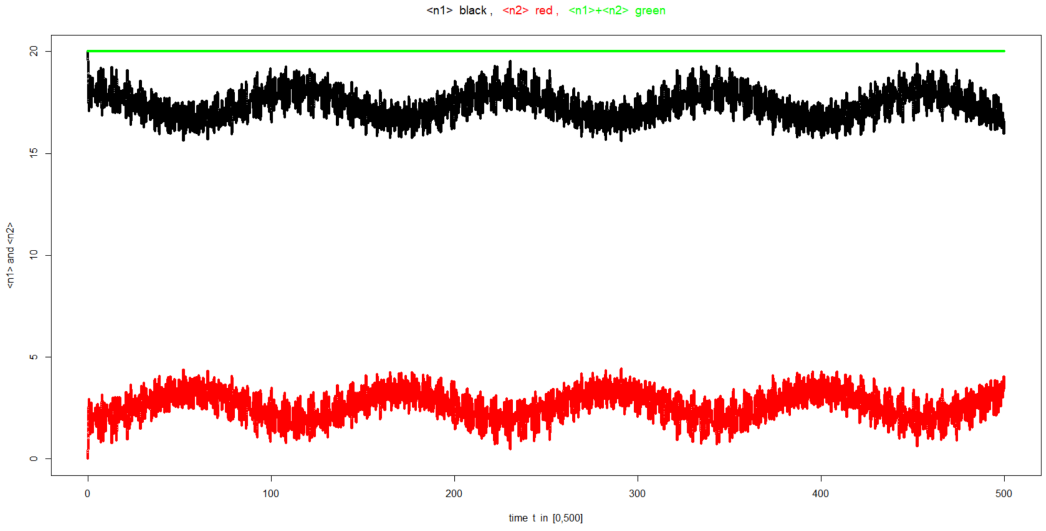
$$g = 3$$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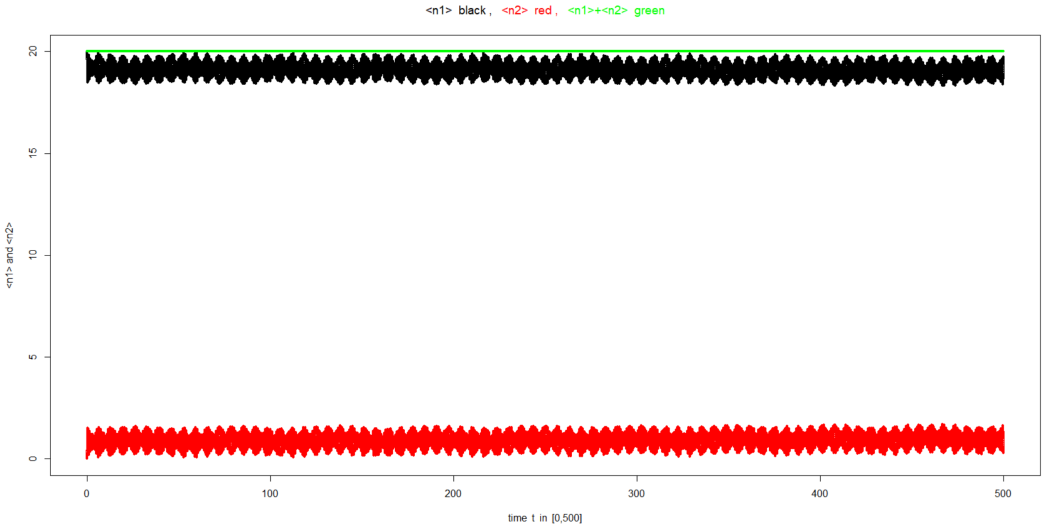
$$g = 4$$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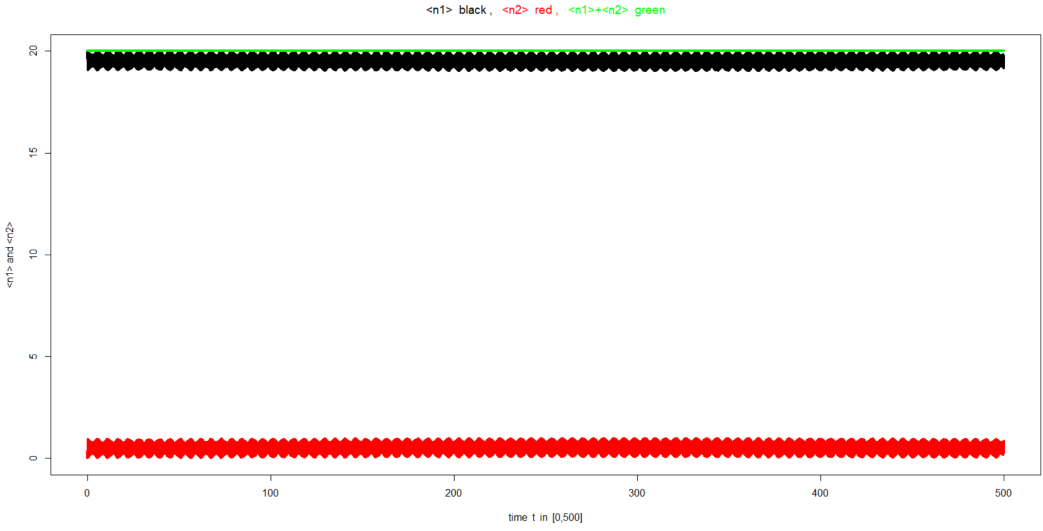
time t in [0,500]

$$g = 5$$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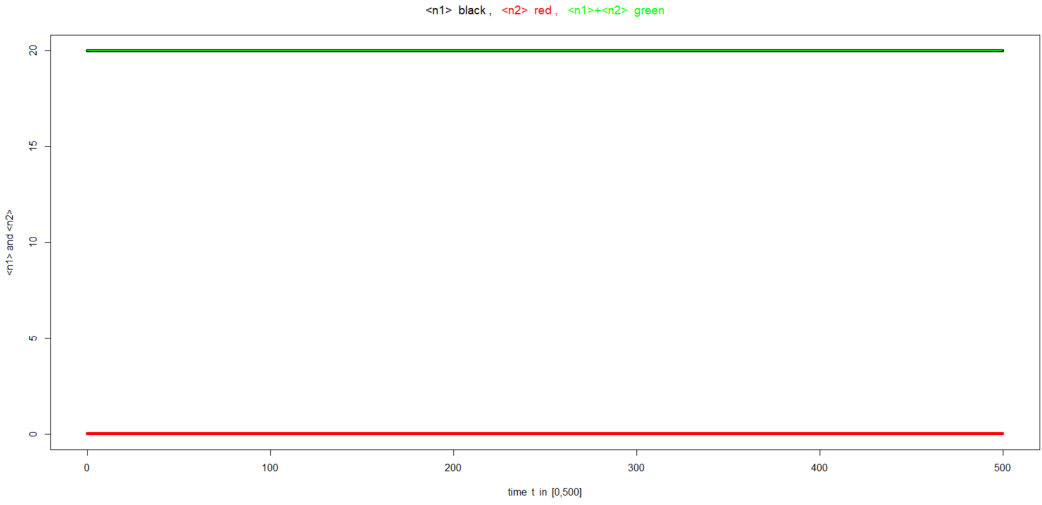
time t in [0,500]

$$g = 25$$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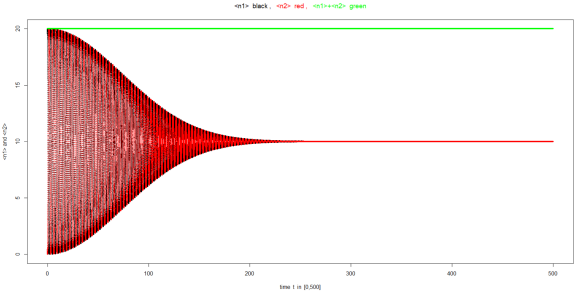
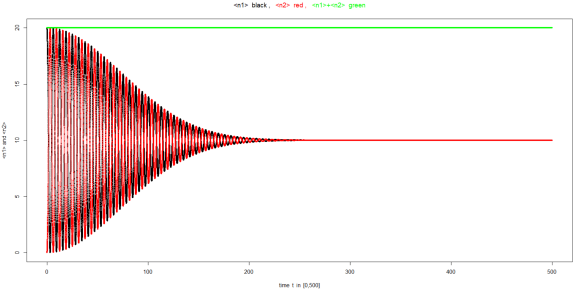
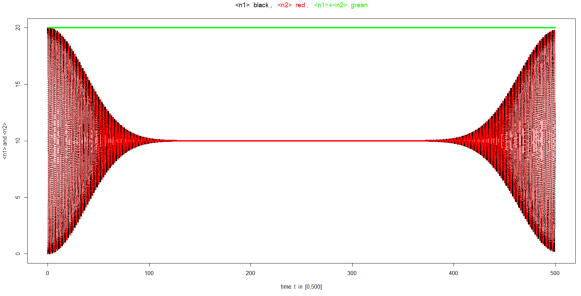


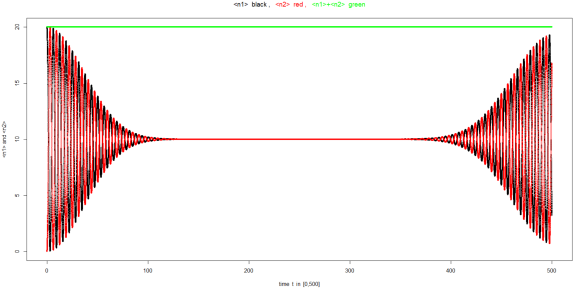
$\varepsilon = 1,\ g = 1/16$



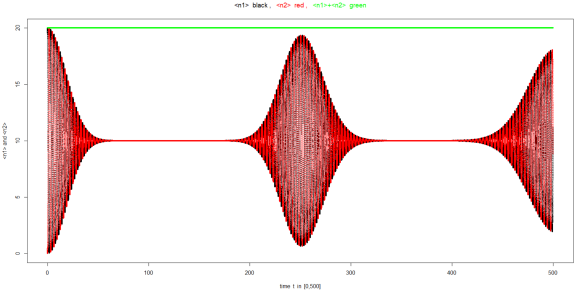
$\varepsilon = 0.5,\ g = 1/16$



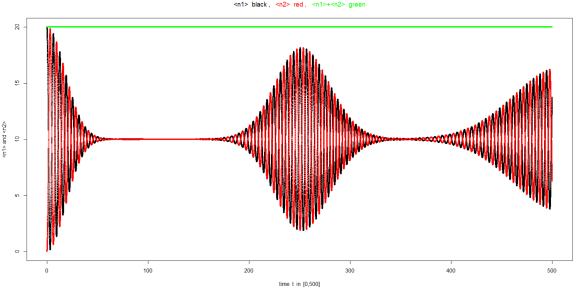
$\varepsilon = 1, \;\; g = 1/8$



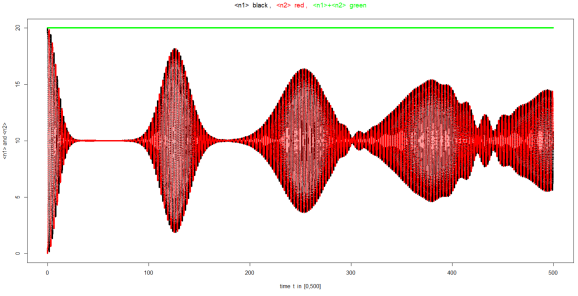
$\varepsilon = 0.5, \;\; g = 1/8$



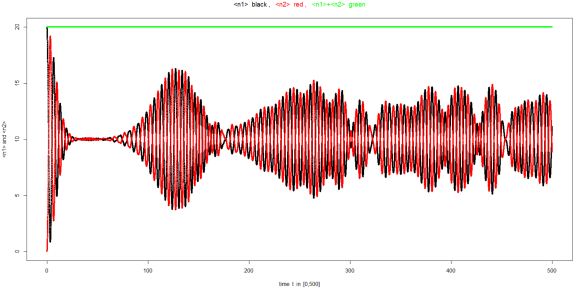
$\varepsilon = 1, \;\; g = 1/4$



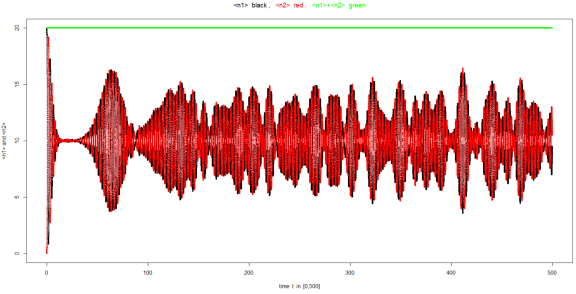
$\varepsilon = 0.5, \;\; g = 1/4$



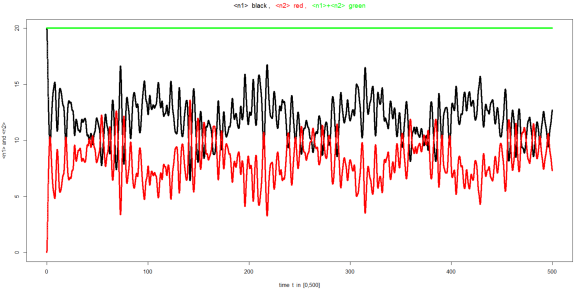
$\varepsilon = 1, \;\; g = 1/2$



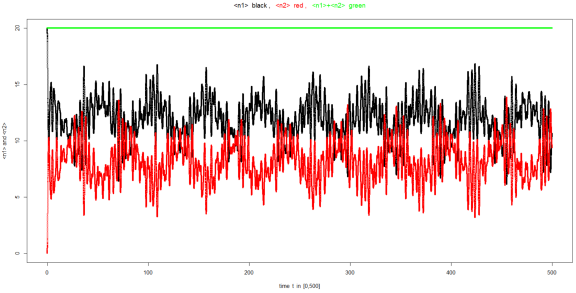
$\varepsilon = 0.5, \;\; g = 1/2$



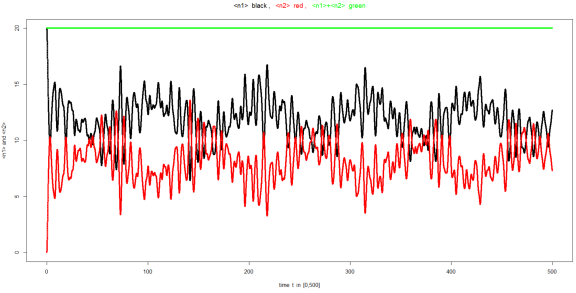
$\varepsilon = 1,\ g = 1, \quad g/\varepsilon = 1$



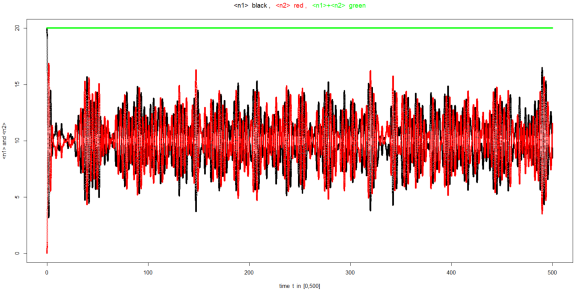
$\varepsilon = 0.5,\ g = 1, \quad \boldsymbol{g/\varepsilon = 2}$



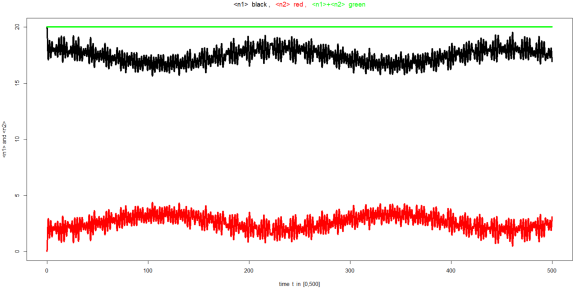
$\varepsilon = 1,\ \boldsymbol{g = 2},\quad \boldsymbol{g/\varepsilon = 2}$



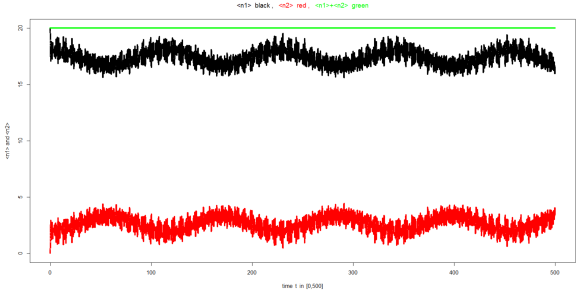
$\varepsilon = 0.5,\ g = 1,\quad \boldsymbol{g/\varepsilon = 2}$



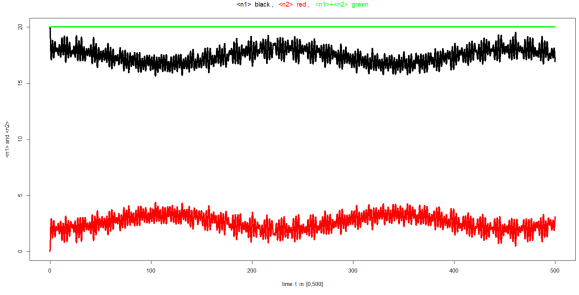
$\varepsilon = 1,\ g = 1.5, \quad g/\varepsilon = 1.5 < 2$



$\varepsilon = 0.5,\ g = 1.5, \quad \mathbf{g/\varepsilon = 3 > 2}$



$\varepsilon = 1,\ \boldsymbol{g = 3.0},\quad \boldsymbol{g/\varepsilon = 3}$



$\varepsilon = 0.5,\ g = 1.5,\quad \boldsymbol{g/\varepsilon = 3}$

1. Setup and Motivation

Central Problem of Many Body Quantum Mechanics:

Calculate Density Matrix Elements or Correlation Functions

$$\langle \psi_t \, , \, a_i^+ a_j \, \psi_t \rangle \, , \qquad \langle \psi_t, a_i^+ a_j^+ a_k \, a_\ell \, \psi_t \rangle \, , \; \ldots$$

this talk: new proposal for doing that

1. Setup and Motivation: Bargmann-Segal Representation

Bargmann-Segal representation:

$$a_j \;=\; \frac{\partial}{\partial z_j}\,, \qquad a_j^+ \;=\; z_j$$

acting on the Hilbert space of analytic functions of $|\Gamma| = L^d$ complex variables:

$$\mathcal{F} \;:=\; \left\{ f = f(\{z_j\}) \,:\, \mathbb{C}^{|\Gamma|} \to \mathbb{C} \;\text{ analytic} \;\middle|\; \|f\|_{\mathcal{F}}^2 = \langle f, f\rangle_{\mathcal{F}} < \infty \right\}$$

with scalar product

$$\langle f, g\rangle_{\mathcal{F}} \;:=\; \int_{\mathbb{C}^{|\Gamma|}=\mathbb{R}^{2|\Gamma|}} f(z)\,\overline{g(z)}\; d\mu(z)$$

$$d\mu(z) \;:=\; \prod_j e^{-|z_j|^2}\; \frac{d\mathrm{Re}z_j\, d\mathrm{Im}z_j}{\pi}$$

1. Setup and Motivation: Bargmann-Segal Representation

Hamiltonian:

$$h \;=\; \sum_{i,j} \varepsilon_{ij}\, z_i \tfrac{\partial}{\partial z_j} \;+\; u \sum_j z_j^2 \tfrac{\partial^2}{\partial z_j^2} \;=:\; h_0 + h_{int}$$

We can allow for a general hopping matrix which should be real and symmetric:

$$\varepsilon \;:=\; (\varepsilon_{ij})_{i,j\in\Gamma} \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}, \qquad \varepsilon_{i,j} = \varepsilon_{j,i}$$

For a nearest neighbor hopping J and on-diagonal trapping potentials ϵ_j:

$$\varepsilon_{ij} \;=\; \begin{cases} -J & \text{if } |i-j| = 1 \\ +\epsilon_j & \text{if } i = j \\ 0 & \text{otherwise } . \end{cases}$$

1. Setup and Motivation: Coherent States and Number States

Coherent and Number states, $\lambda = \{\lambda_j\} \in \mathbb{C}^{|\Gamma|}$:

$$\psi_{coh}(z) \;=\; \psi_{coh}(\,\{z_j\}\,) \;=\; \prod_j \, e^{\lambda_j z_j} \, e^{-\frac{|\lambda_j|^2}{2}} \;=:\; e^{\lambda z} \, e^{-\frac{|\lambda|^2}{2}}$$

$$\psi_{num}(z) \;=\; \psi_{num}(\,\{z_j\}\,) \;=\; \frac{1}{\sqrt{N!}\,|\lambda|^N} \, \big(\textstyle\sum_j \lambda_j z_j\big)^N \;=:\; \frac{(\lambda z)^N}{\sqrt{N!}\,|\lambda|^N}$$

In both cases: $t = 0$ expected number of particles at site j:

$$\langle\, \psi_0 \,,\, a_j^+ a_j \;\psi_0 \,\rangle_{\mathcal{F}} \;=\; |\lambda_j|^2$$

Thus: Total number of particles N:

$$N \;=\; \textstyle\sum_j N_j \;=\; \sum_j |\lambda_j|^2 \;=\; |\lambda|^2$$

2. Overall Strategy and Main Results

2. Overall Strategy and Main Results

1) Time Evolution of States as Fresnel Expectation Values, $t = t_k = k dt$

$$\psi_t(z) \;=\; (e^{-ith}\psi_0)(z) \;=\; \mathrm{E}_\phi[\,\psi_0(U_{\phi,t}z)\,]$$

with unitary evolution matrix

$$U_{k dt}(\phi) \;=\; e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_1)}\cdots e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_k)} \;\in\; \mathbb{C}^{|\Gamma|\times|\Gamma|}$$

2) Correlation Functions as Fresnel Expectation Values

$$\langle\,\psi_t\,,a_i^+a_j\;\psi_t\,\rangle \;=\; \mathrm{E}_\phi\bar{\mathrm{E}}_\theta\big[\;F_{i,j}^{\psi_0}\big(U_{\phi,t}\,,\bar{U}_{\theta,t}\big)\;\big]$$

3) Stochastic Differential Equation (SDE) for Unitary Evolution Matrix $U_{\phi,t}$

$$dU_t \;=\; -i\,U_t\,\big(\,\varepsilon\,dt \;+\; \sqrt{2u}\;dx_t\,\big)\,, \qquad\qquad dx_{kdt} = \sqrt{dt}\,\phi_k$$

2. Overall Strategy and Main Results

4) Correlation Functions from SDEs, $(v_t, \bar{v}_t) := (U_{\phi,t}^T \lambda, \bar{U}_{\theta,t}^T \bar{\lambda})$

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t \,\rangle_{coh/num} \;=\; \mathrm{E}\bar{\mathrm{E}}\big[\; v_{i,t}\; \bar{v}_{j,t}\; P_{\psi_0}(v_t \bar{v}_t)\;\big] \;/\; P_{\psi_0}(v_0 \bar{v}_0)$$

with

$$\begin{aligned} dv_j &= -\,i\,(\varepsilon v)_j\,dt \;-\; i\sqrt{2u}\,v_j\,dx_j \\ d\bar{v}_j &= +\,i\,(\varepsilon \bar{v})_j\,dt \;+\; i\sqrt{2u}\,\bar{v}_j\,dy_j\;, \end{aligned} \qquad P_{coh/num}(x) = e^x \text{ or } x^{N-1}$$

5) Correlation Functions from Girsanov-Transformed SDEs

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t \,\rangle_{coh/num} \;=\; \mathrm{E}\bar{\mathrm{E}}[\; v_{i,t}\; \bar{v}_{j,t}\;]$$

with

$$\begin{aligned} dv_j &= -\,i\,(\varepsilon v)_j\,dt \;-\; i\,2u\,[\log P_{\psi_0}]'(v\bar{v})\; v_j \bar{v}_j\, v_j\; dt \;-\; i\sqrt{2u}\,v_j\,dx_j \\ d\bar{v}_j &= +\,i\,(\varepsilon \bar{v})_j\,dt \;+\; i\,2u\,[\log P_{\psi_0}]'(v\bar{v})\; v_j \bar{v}_j\, \bar{v}_j\; dt \;+\; i\sqrt{2u}\,\bar{v}_j\,dy_j \end{aligned}$$

2. Overall Strategy and Main Results

6) Large N Limit: $g := uN$ fixed, $(w, \bar{w}) := (v, \bar{v})/\sqrt{N}$, exact equation:

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t\,\rangle_{coh/num} \;=\; N \;\times\; \mathrm{E}\bar{\mathrm{E}}[\, w_{i,t}\, \bar{w}_{j,t}\,]$$

with
$$\begin{aligned} dw_j &= -\,i\,(\varepsilon w)_j\,dt \;-\; i\,2g\;[\log P_{\psi_0}]'\; w_j \bar{w}_j\, w_j\;dt \;-\; i\sqrt{2g/N}\, w_j\, dx_j \\ d\bar{w}_j &= +\,i\,(\varepsilon \bar{w})_j\,dt \;+\; i\,2g\;[\log P_{\psi_0}]'\; w_j \bar{w}_j\, \bar{w}_j\;dt \;+\; i\sqrt{2g/N}\, \bar{w}_j\, dy_j \end{aligned}$$

Large N Limit can be read off:

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \ \psi_t\,\rangle_{coh/num} \;=\; N \;\times\; w_i(t)\, \bar{w}_j(t)$$

with $w_j, \bar{w}_j$ given by the ODE system $\quad([\log P_{coh/num}]' \to 1\,)$

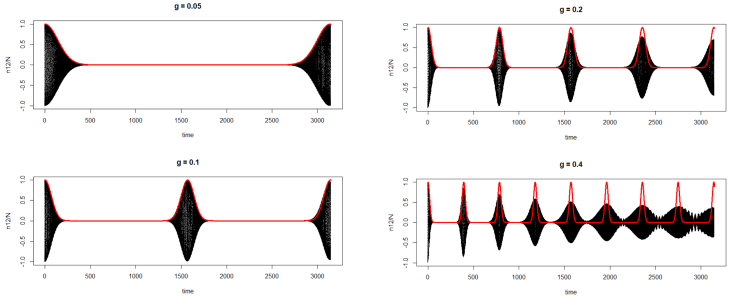
$$\begin{aligned} \dot{w}_j &= -\,i\,(\varepsilon w)_j \;-\; i\,2g\; w_j \bar{w}_j\, w_j \\ \dot{\bar{w}}_j &= +\,i\,(\varepsilon \bar{w})_j \;+\; i\,2g\; w_j \bar{w}_j\, \bar{w}_j \end{aligned}$$

which is the time dependent discrete Gross-Pitaevskii equation.

2. Overall Strategy: Additional Results (appendix has more detail)

- Various Exact PDE Representations (confirming stochastic calculus formalism)
- Collapse and Revivals can be obtained from an approximate ODE System

  Black numbers are from exact diagonalization, the red curve comes from an analytical calculation:



- Equivalence between Mathematical Pendulum and Quartic Double-Well Potential

3. Time Evolution of States as Fresnel Expectation Values

3. Time Evolution of States as Fresnel Expectation Values

Time evolution with Trotter formula, $t = t_k = k\,dt$:

$$e^{-i\,t\,h} \quad = \quad e^{-i\,k\,dt\,(h_0 + h_{int})} \quad \approx \quad \left(e^{-i\,dt\,h_0}\; e^{-i\,dt\,h_{int}} \right)^k$$

Let's consider the action of $e^{-i\,dt\,h_0}$ and $e^{-i\,dt\,h_{int}}$:

3. Time Evolution of States as Fresnel Expectation Values

Action of $e^{-i\,t\,h_0}$:

$$\left(e^{-i\,t\,h_0}\,f\right)(z) \;=\; f\left(e^{-i\,t\,\varepsilon}z\right)$$

since with $z_t := e^{-i\varepsilon t}z$

$$\begin{aligned}\tfrac{\partial}{\partial t}\left[f(z_t)\right] &= \tfrac{\partial}{\partial t}\left[f(e^{-i\,t\,\varepsilon}z)\right] = \textstyle\sum_j \tfrac{\partial z_{t,j}}{\partial t}\,\tfrac{\partial f}{\partial z_j}(z_t) = \sum_j(-i\varepsilon z_t)_j\,\tfrac{\partial f}{\partial z_j}(z_t)\\ &= \left\{-i\textstyle\sum_{j,k}\varepsilon_{j,k}z_k\,\tfrac{\partial f}{\partial z_j}\right\}(z=z_t) = \left\{-i\,h_0\,f\right\}(z_t)\end{aligned}$$

3. Time Evolution of States as Fresnel Expectation Values

Action of $e^{-i\,t\,h_{int}}$:

$$(e^{-i\,t\,h_{int}}\,f)(z) \;=\; \int_{\mathbb{R}^{|\Gamma|}} f(e^{-i\,D_t(\phi)}z)\;\prod_j\, e^{\,i\,\frac{\phi_j^2}{2}}\,\frac{d\phi_j}{\sqrt{2\pi i}} \qquad (1)$$

with $D_t(\phi)$ the $|\Gamma|\times|\Gamma|$ diagonal matrix

$$D_t(\phi) \;=\; \mathrm{diag}\big(\,\{\,\sqrt{2ut}\,\phi_j - ut\,\}_{j\in\Gamma}\,\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

3. Time Evolution of States as Fresnel Expectation Values

Proof of (1):

$$h_{int} \prod_j z_j^{n_j} = \left\{ u \sum_i z_i^2 \frac{\partial^2}{\partial z_i^2} \right\} \prod_j z_j^{n_j} = \left\{ u \sum_i n_i(n_i-1) \right\} \prod_j z_j^{n_j}$$

$$e^{-it\,h_{int}} \prod_j z_j^{n_j} = e^{-iut \sum_j n_j(n_j-1)} \prod_j z_j^{n_j} = \prod_j \left(e^{+iut\,n_j}\, e^{-i\,ut\,n_j^2}\, z_j^{n_j} \right)$$

Now use Fresnel integral ($\sqrt{i} := e^{i\frac{\pi}{4}}$)

$$\int_{\mathbb{R}} e^{-i\lambda\phi}\, e^{i\frac{\phi^2}{2}}\, \frac{d\phi}{\sqrt{2\pi i}} = e^{-i\frac{\lambda^2}{2}}$$

to obtain

$$e^{-it\,h_{int}} \prod_j z_j^{n_j} = \int_{\mathbb{R}^{|\Gamma|}} \prod_j \left(e^{+iut - i\sqrt{2ut}\,\phi_j} z_j \right)^{n_j} \prod_j e^{i\frac{\phi_j^2}{2}}\, \frac{d\phi_j}{\sqrt{2\pi i}} \ . \qquad \blacksquare$$

3. Time Evolution of States as Fresnel Expectation Values

Thus, single Trotter step ($\phi^2 := \sum_j \phi_j^2$) :

$$
\begin{aligned}
(e^{-i\,dt\,h} f)(z) &= \left[e^{-i\,dt\,h_{int}} \left(e^{-i\,dt\,h_0} f\right)\right](z) \\
&= \int_{\mathbb{R}^{|\Gamma|}} f\left(e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi)} z\right)\; e^{i\frac{\phi^2}{2}} \frac{d^{|\Gamma|}\phi}{(2\pi i)^{|\Gamma|/2}}
\end{aligned}
$$

3. Time Evolution of States as Fresnel Expectation Values

Iterating, $t = t_k = kdt$:

$$
\begin{aligned}
(e^{-i\,kdt\,h} f)(z) &= \int_{\mathbb{R}^{k|\Gamma|}} f\big(e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_1)} \cdots e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)}\, z \big) \prod_{\ell=1}^{k} e^{\,i\,\frac{\phi_\ell^2}{2}}\, \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}} \\
&=: \int_{\mathbb{R}^{k|\Gamma|}} f\big(U_{kdt}(\phi)\, z \big)\, dF\big(\{\phi_\ell\}_{\ell=1}^{k} \big) \\
&=: \mathrm{E}\big[\, f(U_{\phi,kdt}\, z) \,\big]
\end{aligned}
$$

with notations ($k, \ell \in \mathbb{N}$ time indices, $j = (j_1, \cdots, j_d) \in \Gamma$ lattice site index)

$$
\begin{aligned}
\phi_\ell &:= (\{\phi_{j,\ell}\}_{j\in\Gamma}) \in \mathbb{R}^{|\Gamma|} \\
\phi_\ell^2 &:= \textstyle\sum_j \phi_{j,\ell}^2 \in \mathbb{R} \\
d^{|\Gamma|}\phi_\ell &:= \textstyle\prod_j d\phi_{j,\ell}
\end{aligned}
$$

3. Time Evolution of States as Fresnel Expectation Values

Theorem 1: Let $h = h_0 + h_{int}$ be the d-dimensional Bose-Hubbard hamiltonian with cubic lattice $\Gamma = \{1, ..., L\}^d$, hopping matrix ε and interaction u. Then, for $t = kdt$, the time evolution of some initial state $\psi_0(z)$ is given by

$$(e^{-i\,kdt\,h}\psi_0)(z) \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \psi_0\big(\,U_{kdt}(\phi)\,z\,\big)\;dF\big(\,\{\phi_\ell\}_{\ell=1}^{k}\,\big) \;=\; \mathsf{E}\big[\;\psi_0\big(\,U_{\phi,kdt}\,z\,\big)\;\big]$$

with unitary evolution matrix

$$U_{kdt}(\phi) \;=\; e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_1)}\cdots e^{-i\,dt\,\varepsilon}e^{-i\,D_{dt}(\phi_k)} \;\in\; \mathbb{C}^{|\Gamma|\times|\Gamma|}$$

$$D_{dt}(\phi_\ell) \;=\; \mathrm{diag}\big(\,\{\,\sqrt{2u\,dt}\;\phi_{j,\ell} \;-\; u\,dt\,\}_{j\in\Gamma}\,\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

and Fresnel measure given by

$$\mathsf{E}\big[\,\cdot\,\big] \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \cdot\; dF\big(\,\{\phi_\ell\}_{\ell=1}^{k}\,\big) \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \cdot\; \prod_{\ell=1}^{k} e^{\,i\,\frac{\phi_\ell^2}{2}}\,\frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}}$$

4. Correlation Functions as Fresnel Expectation Values

4. Correlation Functions as Fresnel Expectation Values

Correlation Functions in Bargmann-Segal Representation:

$$\begin{aligned} \langle\, \psi_t \,,\, a_i^+ a_j \;\; \psi_t \,\rangle &= \langle\, \psi_t \,,\, [a_j \;\; a_i^+ - \delta_{i,j}]\, \psi_t \,\rangle \\ &= \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar{z}_i \; |\psi_t(z)|^2 \, d\mu(z) \;\; - \;\; \delta_{i,j} \end{aligned}$$

From Theorem 1,

$$\begin{aligned} \psi_{kdt}(z) &= \int_{\mathbb{R}^{k|\Gamma|}} \psi_0(\, U_{\phi,kdt} z \,)\; dF(\phi) \\ \overline{\psi_{kdt}(z)} &= \int_{\mathbb{R}^{k|\Gamma|}} \overline{\psi_0(\, U_{\theta,kdt} z \,)}\; d\bar{F}(\theta) \end{aligned}$$

with

$$dF(\phi) \;=\; \prod_{\ell=1}^{k} e^{+\,i\frac{\varphi_\ell^2}{2}} \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}} \quad , \qquad d\bar{F}(\theta) \;=\; \prod_{\ell=1}^{k} e^{-\,i\frac{\theta_\ell^2}{2}} \frac{d^{|\Gamma|}\theta_\ell}{[2\pi(-i)]^{|\Gamma|/2}}$$

4. Correlation Functions as Fresnel Expectation Values

Thus, with $t = kdt$,

$$\langle \psi_t \, , a_j a_i^+ \psi_t \rangle \;=\; \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar{z}_i \, |\psi_t(z)|^2 \, d\mu(z)$$

$$= \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar{z}_i \int_{\mathbb{R}^{k|\Gamma|}} \psi_0(U_{\phi,t} z) \, dF(\phi) \int_{\mathbb{R}^{k|\Gamma|}} \overline{\psi_0(U_{\theta,t} z)} \, d\bar{F}(\theta) \;\; d\mu(z)$$

$$= \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \left\{ \int_{\mathbb{C}^{|\Gamma|}} z_j \, \bar{z}_i \, \psi_0(U_{\phi,t} z) \, \overline{\psi_0(U_{\theta,t} z)} \, d\mu(z) \right\} dF(\phi) \, d\bar{F}(\theta)$$

The red integral is the expectation over the bosonic Fock space and can be calculated:

4. Correlation Functions as Fresnel Expectation Values

The integrand is:

For coherent states:

$$\begin{aligned} z_j\,\bar{z}_i\;\psi_0(U_{\phi,t}z)\;\overline{\psi_0(U_{\theta,t}z)} &= z_j\,\bar{z}_i\;\exp\{\,\lambda\cdot U_{\phi,t}z\,\}\;\exp\{\,\bar{\lambda}\cdot\bar{U}_{\theta,t}\bar{z}\,\}\;e^{-|\lambda|^2} \\ &= z_j\,\bar{z}_i\;\exp\{\,U^T_{\phi,t}\lambda\cdot z\,\}\;\exp\{\,\bar{U}^T_{\theta,t}\bar{\lambda}\cdot\bar{z}\,\}\;e^{-|\lambda|^2} \end{aligned}$$

For number states:

$$\begin{aligned} z_j\,\bar{z}_i\;\psi_0(U_{\phi,t}z)\;\overline{\psi_0(U_{\theta,t}z)} &= z_j\,\bar{z}_i\;\tfrac{1}{N!N^N}\;\left(\lambda\cdot U_{\phi,t}z\right)^N\left(\bar{\lambda}\cdot\bar{U}_{\theta,t}\bar{z}\right)^N \\ &= z_j\,\bar{z}_i\;\tfrac{1}{N!N^N}\;\left(U^T_{\phi,t}\lambda\cdot z\right)^N\left(\bar{U}^T_{\theta,t}\bar{\lambda}\cdot\bar{z}\right)^N \end{aligned}$$

4. Correlation Functions as Fresnel Expectation Values

Using (with $(\lambda, \bar{\lambda}) \to (U^T_{\phi,t}\lambda\,,\, \bar{U}^T_{\theta,t}\bar{\lambda})$ and recall $d\mu(z) = \prod_j e^{-|z_j|^2} \frac{d\mathrm{Re}z_j\, d\mathrm{Im}z_j}{\pi}$)

$$\int_{\mathbb{C}^{|\Gamma|}} e^{\lambda z + \bar{\lambda}\bar{z}}\, d\mu(z) \;=\; e^{\lambda\bar{\lambda}}$$

$$\int_{\mathbb{C}^{|\Gamma|}} z_j\,\bar{z}_i\; e^{\lambda z + \bar{\lambda}\bar{z}}\, d\mu(z) \;=\; \frac{\partial}{\partial\lambda_j}\frac{\partial}{\partial\bar{\lambda}_i}\, e^{\lambda\bar{\lambda}} \;=\; (\lambda_i\,\bar{\lambda}_j + \delta_{i,j})\, e^{\lambda\bar{\lambda}}$$

and

$$\tfrac{1}{N!}\int_{\mathbb{C}^{|\Gamma|}} (\lambda z)^N\,(\bar{\lambda}\bar{z})^N\, d\mu(z) \;=\; (\lambda\bar{\lambda})^N$$

$$\tfrac{1}{N!}\int_{\mathbb{C}^{|\Gamma|}} z_j\,\bar{z}_i\;(\lambda z)^N\,(\bar{\lambda}\bar{z})^N\, d\mu(z) \;=\; \frac{\partial}{\partial\lambda_j}\frac{\partial}{\partial\bar{\lambda}_i}\,\frac{(\lambda\bar{\lambda})^{N+1}}{N+1} \;=\; N\,\lambda_i\,\bar{\lambda}_j\,(\lambda\bar{\lambda})^{N-1} \;+\; \delta_{i,j}\,(\,\lambda\bar{\lambda}\,)^N$$

we find

4. Correlation Functions as Fresnel Expectation Values

Theorem 2: For $t = kdt$ let $\mathrm{E}\bar{\mathrm{E}}[\,\cdot\,] = \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \cdot\; dF(\phi)\, d\bar{F}(\theta)$. Then:

a) For an arbitrary initial state ψ_0,

$$\psi_t(z) \;=\; (e^{-ith}\psi_0)(z) \;=\; \mathrm{E}[\,\psi_0(U_{\phi,t}z)\,] \qquad \text{(reminder, same as Theorem 1)}$$

b) For a coherent state $\psi_0(z) = e^{\lambda z} e^{-|\lambda|^2/2}$,

$$\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle \;=\; \mathrm{E}\bar{\mathrm{E}}\Big[\; [U_{\phi,t}^T\lambda]_i\; [\bar{U}_{\theta,t}^T\bar{\lambda}]_j\; \exp\{\, U_{\phi,t}^T\lambda \cdot \bar{U}_{\theta,t}^T\bar{\lambda}\,\}\;\Big] \,/\, e^{\lambda\bar{\lambda}}$$

c) For a number state $\psi_0(z) = (\lambda z/|\lambda|)^N/\sqrt{N!}$,

$$\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle \;=\; \mathrm{E}\bar{\mathrm{E}}\Big[\; [U_{\phi,t}^T\lambda]_i\; [\bar{U}_{\theta,t}^T\bar{\lambda}]_j\; \big(\, U_{\phi,t}^T\lambda \cdot \bar{U}_{\theta,t}^T\bar{\lambda}\,\big)^{N-1}\;\Big] \,/\, (\lambda\bar{\lambda})^{N-1}$$

5. SDE Representation

5. SDE Representation: Reminder Brownian Motion and Wiener Measure

Standard Brownian motion in discrete time $t = t_k = kdt$:

$$x_{t_k} \;=\; \sqrt{dt}\,\sum_{\ell=1}^{k}\phi_\ell \qquad\qquad dx_{t_k} := x_{t_k} - x_{t_{k-1}} = \sqrt{dt}\,\phi_k$$

Wiener measure in discrete time with fixed time horizon $T = t_n = ndt$:

$$dW \;=\; \prod_{\ell=1}^{n} e^{-\frac{\phi_\ell^2}{2}}\,\frac{d\phi_\ell}{\sqrt{2\pi}} \;=\; \prod_{\ell=1}^{n} e^{-\frac{(x_{t_\ell}-x_{t_{\ell-1}})^2}{2\,dt}}\,\frac{dx_{t_\ell}}{\sqrt{2\pi\,dt}}$$

Brownian motion calculation rule, basic to stochastic calculus:

$$(dx_t)^2 \;=\; dt$$

For more background, see the appendix compact summary stochastic calculus in

`https://arxiv.org/pdf/2205.02010.pdf`

5. SDE Representation: Fresnel Brownian Motion and Fresnel Measure

Then Fresnel Brownian motion is

$$x_{t_k} \;=\; \sqrt{dt}\,\sum_{\ell=1}^{k}\phi_\ell \qquad\qquad dx_{t_k} := x_{t_k} - x_{t_{k-1}} = \sqrt{dt}\,\phi_k$$

with the ϕ_ℓ to be integrated against Fresnel measure,

$$dF \;=\; \prod_{\ell=1}^{n} e^{i\frac{\phi_\ell^2}{2}}\,\frac{d\phi_\ell}{\sqrt{2\pi i}} \;=\; \prod_{\ell=1}^{n} e^{i\,\frac{(x_{t_\ell}-x_{t_{\ell-1}})^2}{2\,dt}}\,\frac{dx_{t_\ell}}{\sqrt{2\pi i\,dt}}$$

Fresnel Brownian motion calculation rule:

$$(dx_t)^2 \;=\; i\,dt$$

For more background, see the appendix compact summary stochastic calculus in

`https://arxiv.org/pdf/2205.02010.pdf`

5. SDE Representation: Unitary Evolution Matrix U_{kdt}

Unitary evolution matrix of Theorem 1,

$$U_{kdt} \;=\; e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_1)} \cdots e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)} \;=\; U_{(k-1)dt}\;\; e^{-i\,dt\,\varepsilon} e^{-i\,D_{dt}(\phi_k)}$$

with diagonal matrix $D_{dt}(\phi_k)$ given by

$$\begin{aligned} D_{dt}(\phi_k) &= \mathrm{diag}\big(\,\{\,\sqrt{2u}\,\sqrt{dt}\;\phi_{j,k} \;-\; u\,dt\,\}_{j\in\Gamma}\,\big) \\ &= \mathrm{diag}\big(\,\{\,\sqrt{2u}\;dx_{j,kdt} \;-\; u\,dt\,\}_{j\in\Gamma}\,\big) \\ &=: \;\sqrt{2u}\;dx_{kdt} \;-\; u\,dt\;Id \end{aligned}$$

and diagonal matrix of Fresnel Brownian motions

$$dx_{kdt} \;:=\; \mathrm{diag}\big(\,\{\,dx_{j,kdt}\,\}_{j\in\Gamma}\,\big) \;\in\; \mathbb{R}^{|\Gamma|\times|\Gamma|}$$

which satisfies the matrix equation $(dx_{kdt})^2 \;=\; i\,dt\;Id$.

5. SDE Representation: Unitary Evolution Matrix U_{kdt}

Up to terms $O(dt^{3/2})$,

$$
\begin{aligned}
e^{-i\,D_{dt}(\phi_k)} &= 1 \;-\; i\,D_{dt}(\phi_k) \;-\; \tfrac{1}{2}\,[D_{dt}(\phi_k)]^2 \\
&= 1 \;-\; i\,\big[\,\sqrt{2u}\,dx_{kdt} \;-\; u\,dt\,Id\,\big] \;-\; \tfrac{1}{2}\,\big[\,\sqrt{2u}\,dx_{kdt} \;-\; u\,dt\,Id\,\big]^2 \\
&= 1 \;-\; i\sqrt{2u}\,dx_{kdt} \;+\; i\,u\,dt\,Id \;-\; \tfrac{1}{2}\,\big[\,\sqrt{2u}\,dx_{kdt}\,\big]^2 \\
&= 1 \;-\; i\sqrt{2u}\,dx_{kdt} \;+\; i\,u\,dt\,Id \;-\; i\,u\,dt\,Id \\
&= 1 \;-\; i\sqrt{2u}\,dx_{kdt}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U_{kdt} &= U_{(k-1)dt}\;\; e^{-i\,dt\,\varepsilon}\,e^{-i\,D_{dt}(\phi_k)} \\
&= U_{(k-1)dt}\,\big(\,1 \;-\; i\,dt\,\varepsilon\,\big)\,\big(\,1 \;-\; i\sqrt{2u}\;dx_{kdt}\,\big) \\
&= U_{(k-1)dt}\,\big(\,1 \;-\; i\,dt\,\varepsilon \;-\; i\sqrt{2u}\;dx_{kdt}\,\big)
\end{aligned}
$$

$$
\Rightarrow \qquad dU_{t_k} \;:=\; U_{t_k} - U_{t_{k-1}} \;=\; -i\,U_{t_{k-1}}\,\big(\,dt\,\varepsilon \;+\; \sqrt{2u}\;dx_{t_k}\,\big)
$$

5. SDE Representation

Theorem 3: a) The unitary evolution matrix U_t satisfies the SDE

$$dU_t \;=\; -i\,U_t\,\big(\,\varepsilon\,dt \;+\; \sqrt{2u}\,dx_t\,\big)$$

b) The correlation functions have the representations

$$\langle\,\psi_t\,,\,a_i^+ a_j\;\psi_t\,\rangle_{coh} \;=\; \mathrm{E}\bar{\mathrm{E}}\big[\;v_i\,\bar{v}_j\;e^{v\bar{v}}\;\big]\;/\;e^{\lambda\bar{\lambda}}$$

$$\langle\,\psi_t\,,\,a_i^+ a_j\;\psi_t\,\rangle_{num} \;=\; \mathrm{E}\bar{\mathrm{E}}\big[\;v_i\,\bar{v}_j\;(v\bar{v})^{N-1}\;\big]\;/\;(\lambda\bar{\lambda})^{N-1}$$

with $v, \bar{v} \in \mathbb{C}^{|\Gamma|}$, $v = (v_{j,x,t})$, $\bar{v} = (\bar{v}_{j,y,t})$, given by the SDE system

$$\begin{aligned} dv_j &= -\,i\,dt\,(\varepsilon v)_j \;-\; i\,\sqrt{2u}\,v_j\,dx_j \\ d\bar{v}_j &= +\,i\,dt\,(\varepsilon \bar{v})_j \;+\; i\,\sqrt{2u}\,\bar{v}_j\,dy_j \end{aligned}$$

with initial conditions $(v_{x,0}, \bar{v}_{y,0}) = (\lambda, \bar{\lambda})$ and $\mathrm{E}\bar{\mathrm{E}} = \mathrm{E}_x\bar{\mathrm{E}}_y$.

5. SDE Representation

Proof 3b) From Theorem 2,

$$\begin{aligned}
\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle_{coh} &= \mathrm{E}\bar{\mathrm{E}}\Big[\; [U_{\phi,t}^T\lambda]_i\; [\bar{U}_{\theta,t}^T\bar{\lambda}]_j\;\; \exp\{\; U_{\phi,t}^T\lambda \cdot \bar{U}_{\theta,t}^T\bar{\lambda}\;\}\;\Big] \;/\; e^{\lambda\bar{\lambda}} \\
\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle_{num} &= \mathrm{E}\bar{\mathrm{E}}\Big[\; [U_{\phi,t}^T\lambda]_i\; [\bar{U}_{\theta,t}^T\bar{\lambda}]_j\;\; \big(\; U_{\phi,t}^T\lambda \cdot \bar{U}_{\theta,t}^T\bar{\lambda}\big)^{N-1}\;\Big] \;/\; (\lambda\bar{\lambda})^{N-1}
\end{aligned}$$

With $(\, x_{kdt} = \sqrt{dt}\sum_{j=1}^k \phi_j\,,\;\; y_{kdt} = \sqrt{dt}\sum_{j=1}^k \theta_j\,)$

$$\begin{aligned}
v \;=\; v_{x,t} \;&:=\; U_{x,t}^T\lambda \;\in\; \mathbb{C}^{|\Gamma|} \\
\bar{v} \;=\; \bar{v}_{y,t} \;&:=\; \bar{U}_{y,t}^T\bar{\lambda} \;\in\; \mathbb{C}^{|\Gamma|}
\end{aligned}$$

this looks as follows

$$\begin{aligned}
\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle_{coh} &= \mathrm{E}\bar{\mathrm{E}}\big[\; v_i\, \bar{v}_j\; e^{v\bar{v}}\;\big] \;/\; e^{\lambda\bar{\lambda}} \\
\langle\, \psi_t\,,\, a_i^+ a_j\, \psi_t \,\rangle_{num} &= \mathrm{E}\bar{\mathrm{E}}\big[\; v_i\, \bar{v}_j\; (v\bar{v})^{N-1}\;\big] \;/\; (\lambda\bar{\lambda})^{N-1}
\end{aligned}$$

5. SDE Representation

Since

$$dU_t \;=\; -i\,U_t\left(\varepsilon\,dt \;+\; \sqrt{2u}\,dx_t\right)$$

and because of $\varepsilon^T = \varepsilon$, $dx_t^T = dx_t$, we obtain

$$\begin{aligned} dU_t^T &= -i\left(\varepsilon\,dt \;+\; \sqrt{2u}\,dx_t\right)U_t^T \\ d\bar{U}_t^T &= +i\left(\varepsilon\,dt \;+\; \sqrt{2u}\,dy_t\right)\bar{U}_t^T \end{aligned}$$

Thus, with $v_t = U_t^T\lambda$, $\bar{v}_t = \bar{U}_t^T\bar{\lambda}$,

$$\begin{aligned} dv &= -\,i\left(dt\,\varepsilon \;+\; \sqrt{2u}\,dx_t\right)v \\ d\bar{v} &= +\,i\left(dt\,\varepsilon \;+\; \sqrt{2u}\,dy_t\right)\bar{v} \end{aligned}$$

with initial conditions $v_0 = U_0^T\lambda = \lambda$, $\bar{v}_0 = \bar{U}_0^T\bar{\lambda} = \bar{\lambda}$. ■

6. Girsanov Transformed SDE Representation

6.1 Unitary Time Evolution as a Martingale
6.2 Girsanov Transformed SDE System

6.1 Unitary Time Evolution as a Martingale

6.1 Unitary Time Evolution as a Martingale

As in Theorem 3, one finds just for the norms $\|\psi_t\|^2$:

$$\begin{aligned} \langle \psi_t, \psi_t \rangle_{coh} &= \mathrm{E}\bar{\mathrm{E}}[\, e^{v\bar{v}}\,] \,/\, e^{\lambda\bar{\lambda}} \\ \langle \psi_t, \psi_t \rangle_{num} &= \mathrm{E}\bar{\mathrm{E}}[\, (v\bar{v})^{N-1}] \,/\, (\lambda\bar{\lambda})^{N-1} \end{aligned}$$

with $v\bar{v} = \sum_j v_j \bar{v}_j$ and $v_j, \bar{v}_j$ given by the SDE system

$$\begin{aligned} dv_j &= -\, i\, dt\, (\varepsilon v)_j \;-\; i\,\sqrt{2u}\, v_j\, dx_j \\ d\bar{v}_j &= +\, i\, dt\, (\varepsilon \bar{v})_j \;+\; i\,\sqrt{2u}\, \bar{v}_j\, dy_j \end{aligned}$$

These quantities have to be independent of time, we have to have

$$\begin{aligned} \mathrm{E}\bar{\mathrm{E}}[\, e^{v\bar{v}}\,] &= e^{v_0\bar{v}_0} = e^{\lambda\bar{\lambda}} \\ \mathrm{E}\bar{\mathrm{E}}[\, (v\bar{v})^{N-1}] &= (v_0\bar{v}_0)^{N-1} = (\lambda\bar{\lambda})^{N-1} \end{aligned}$$

How can this be understood in an SDE context?

6.1 Unitary Time Evolution as a Martingale

From

$$\begin{aligned} dv &= -i\,(\varepsilon\,dt + \sqrt{2u}\,dx_t)\,v \\ d\bar{v} &= +i\,(\varepsilon\,dt + \sqrt{2u}\,dy_t)\,\bar{v} \end{aligned}$$

we get

$$dv^T = -i\,v^T(\varepsilon\,dt + \sqrt{2u}\,dx_t)\ ,$$

$$\begin{aligned} d(v\bar{v}) \overset{notation}{=} d(v^T\bar{v}) &= dv^T\bar{v} + v^T d\bar{v} + dv^T d\bar{v} \\ &= -i\,v^T(\varepsilon\,dt + \sqrt{2u}\,dx_t)\,\bar{v} + i\,v^T(\varepsilon\,dt + \sqrt{2u}\,dy_t)\,\bar{v} + 0 \\ &= -i\,v^T\varepsilon\,\bar{v}\,dt + i\,v^T\varepsilon\,\bar{v}\,dt - i\sqrt{2u}\,v^T(dx_t - dy_t)\,\bar{v} \\ &= -i\sqrt{2u}\,\textstyle\sum_j v_j\bar{v}_j\,(dx_{j,t} - dy_{j,t}) \end{aligned}$$

The quantity $v\bar{v}$ is a martingale, its $d(v\bar{v})$ has no drift part. Since $\mathrm{E}[dx_{j,t}] = \bar{\mathrm{E}}[dy_{j,t}] = 0$,

$$\Rightarrow \qquad \mathrm{E}\bar{\mathrm{E}}[\,(v\bar{v})_{t_k}\,] = (v\bar{v})_0 + \textstyle\sum_{\ell=1}^{k} \mathrm{E}\bar{\mathrm{E}}[\,d(v\bar{v})_{t_\ell}\,] = (v\bar{v})_0$$

6.1 Unitary Time Evolution as a Martingale

Ok, but we need more, we need $\mathrm{E}\bar{\mathrm{E}}[\, f(v_t \bar{v}_t)\,] = f(v_0 \bar{v}_0)$ for arbitrary f. With Ito-Lemma:

$$df(v_t \bar{v}_t) \;=\; f'(v\bar{v})\, d(v_t \bar{v}_t) \;+\; \tfrac{1}{2}\, f''(v\bar{v})\, [d(v_t \bar{v}_t)]^2$$

Now,

$$\begin{aligned}
[d(v_t \bar{v}_t)]^2 &= \Big\{ -\, i\sqrt{2u}\, \textstyle\sum_j v_j \bar{v}_j\, (dx_{j,t} - dy_{j,t}) \Big\}^2 \\
&= -\, 2u \sum_{i,j} v_i \bar{v}_i\; v_j \bar{v}_j\; (dx_{i,t} - dy_{i,t})(dx_{j,t} - dy_{j,t}) \\
&= 0
\end{aligned}$$

since $dx_{i,t} dx_{j,t} = dy_{i,t} dy_{j,t} = dx_{i,t} dy_{j,t} = 0$ for $i \neq j$ and for $i = j$:

$$\begin{aligned}
(dx_{j,t} - dy_{j,t})^2 &= (dx_{j,t})^2 \;-\; 2\, dx_{j,t} dy_{j,t} \;+\; (dy_{j,t})^2 \\
&= \;+\; i\, dt \;-\; 2 \cdot 0 \;-\; i\, dt \;=\; 0
\end{aligned}$$

6.1 Unitary Time Evolution as a Martingale

Thus,

$$\begin{aligned} df(v_t\bar{v}_t) &= f'(v\bar{v})\, d(v_t\bar{v}_t) + \tfrac{1}{2} f''(v\bar{v}) \underbrace{[d(v_t\bar{v}_t)]^2}_{=0} \\ &= -i\sqrt{2u}\, f'(v_t\bar{v}_t) \sum_j v_j\bar{v}_j\, (dx_{j,t} - dy_{j,t}) \end{aligned} \tag{2}$$

has no drift part which gives $\mathrm{E}\bar{\mathrm{E}}[\, df(v_t\bar{v}_t)\,] = 0$ and therefore

$$\mathrm{E}\bar{\mathrm{E}}[\, f(v_t\bar{v}_t)\,] = f(v_0\bar{v}_0)\ .$$

Equation (2) allows us in the next section 6.2 to absorb the the quantities $e^{v\bar{v}}$ or $(v\bar{v})^{N-1}$ into the Fresnel integration measure and to arrive at the compact expressions

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t\,\rangle_{coh/num} = \mathrm{E}\bar{\mathrm{E}}[\ v_i\ \bar{v}_j\ ]$$

with new, and initial state *coh*/*num* dependent, SDE's for the $v_i, \bar{v}_j$. Result is Theorem 4.

6.2 Girsanov Transformed SDE System

6.2 Girsanov Transformed SDE System

Untransformed representation:

$$\begin{aligned}\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t\,\rangle_{coh} &= \mathrm{E}\bar{\mathrm{E}}\big[\ v_i\,\bar{v}_j\ e^{v\bar{v}}\ \big]\ /\ e^{\lambda\bar{\lambda}} \\ \langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t\,\rangle_{num} &= \mathrm{E}\bar{\mathrm{E}}\big[\ v_i\,\bar{v}_j\ (v\bar{v})^{N-1}\ \big]\ /\ (\lambda\bar{\lambda})^{N-1}\end{aligned}$$

We write

$$\langle\, \psi_t\,,\, a_i^+ a_j\ \psi_t\,\rangle \;=\; \mathrm{E}\bar{\mathrm{E}}\big[\ v_{i,t}\,\bar{v}_{j,t}\ P(v_t\bar{v}_t)\ \big]\ /\ P(v_0\bar{v}_0)$$

Now, with $p := \log P$,

$$\begin{aligned}P(v_{t_k}\bar{v}_{t_k})\ /\ P(v_0\bar{v}_0) &= \exp\big\{\ p(v_{t_k}\bar{v}_{t_k}) - p(v_0\bar{v}_0)\ \big\} = \exp\Big\{\ \textstyle\sum_{\ell=1}^k dp(v_{t_\ell}\bar{v}_{t_\ell})\ \Big\} \\ &\overset{(2)}{=} \exp\Big\{ - i\sqrt{2u}\ \textstyle\sum_{\ell=1}^k \sum_j [p'(v\bar{v})(v_j\bar{v}_j)]_{t_{\ell-1}}\,(dx_{j,t_\ell} - dy_{j,t_\ell})\ \Big\} \\ &= \exp\Big\{ - i\sqrt{2u\,dt}\ \textstyle\sum_{\ell=1}^k \sum_j [p'(v\bar{v})(v_j\bar{v}_j)]_{t_{\ell-1}}\,(\phi_{j,\ell} - \theta_{j,\ell})\ \Big\}\end{aligned}$$

6.2 Girsanov Transformed SDE System

Since

$$\mathrm{E}\bar{\mathrm{E}}[\,\cdot\,] \;=\; \int_{\mathbb{R}^{k|\Gamma|}} \int_{\mathbb{R}^{k|\Gamma|}} \;\cdot\; dF(\phi) d\bar{F}(\theta)$$

with

$$dF(\phi) \;=\; \prod_{\ell=1}^{k} e^{+i\frac{\phi_\ell^2}{2}} \frac{d^{|\Gamma|}\phi_\ell}{(2\pi i)^{|\Gamma|/2}} \;, \qquad d\bar{F}(\theta) \;=\; \prod_{\ell=1}^{k} e^{-i\frac{\theta_\ell^2}{2}} \frac{d^{|\Gamma|}\theta_\ell}{[2\pi(-i)]^{|\Gamma|/2}}$$

the quantity

$$P(v_{t_k}\bar{v}_{t_k}) \;/\; P(v_0\bar{v}_0) \;=\; \exp\Big\{ -i\sqrt{2u\,dt}\, \sum_{\ell=1}^{k} \sum_j \left[p'(v\bar{v})(v_j\bar{v}_j)\right]_{t_{\ell-1}} (\phi_{j,\ell} - \theta_{j,\ell}) \Big\}$$

can be absorbed into the Fresnel measure by completing the square. In stochastic calculus, this is called a Girsanov transformation:

$$\begin{aligned} \tilde{\phi}_{j,\ell} &:= \phi_{j,\ell} - \sqrt{2u\,dt}\,[\,p'(v\bar{v})\,v_j\bar{v}_j\,]_{(\ell-1)dt} \\ \tilde{\theta}_{j,\ell} &:= \theta_{j,\ell} - \sqrt{2u\,dt}\,[\,p'(v\bar{v})\,v_j\bar{v}_j\,]_{(\ell-1)dt} \end{aligned}$$

6.2 Girsanov Transformed SDE System

or equivalently

$$
\begin{aligned}
d\tilde{x}_{j,t_\ell} &:= dx_{j,t_\ell} - \sqrt{2u}\, dt\, [\, p'(v\bar{v})\, v_j \bar{v}_j\,]_{(\ell-1)dt} \\
d\tilde{y}_{j,t_\ell} &:= dy_{j,t_\ell} - \sqrt{2u}\, dt\, [\, p'(v\bar{v})\, v_j \bar{v}_j\,]_{(\ell-1)dt}
\end{aligned}
$$

We get a new SDE system:

$$
\begin{aligned}
dv_j &= -\, i\, dt\, (\varepsilon v)_j - i\sqrt{2u}\, v_j\, dx_j \\
&= -\, i\, dt\, (\varepsilon v)_j - i\sqrt{2u}\, v_j \left[d\tilde{x}_j + \sqrt{2u}\, dt\, p' v_j \bar{v}_j \right] \\
&= -\, i\, dt\, (\varepsilon v)_j - i\, 2u\, dt\, p' v_j \bar{v}_j\, v_j - i\sqrt{2u}\, v_j\, d\tilde{x}_j
\end{aligned}
$$

$$
\begin{aligned}
d\bar{v}_j &= +\, i\, dt\, (\varepsilon \bar{v})_j + i\sqrt{2u}\, \bar{v}_j\, dy_j \\
&= +\, i\, dt\, (\varepsilon \bar{v})_j + i\sqrt{2u}\, \bar{v}_j \left[d\tilde{y}_j + \sqrt{2u}\, dt\, p' v_j \bar{v}_j \right] \\
&= +\, i\, dt\, (\varepsilon \bar{v})_j + i\, 2u\, dt\, p' v_j \bar{v}_j\, \bar{v}_j + i\sqrt{2u}\, \bar{v}_j\, d\tilde{y}_j
\end{aligned}
$$

6.2 Girsanov Transformed SDE System

Theorem 4: The correlation functions of Theorem 3 have the following equivalent representation:

$$\langle\, \psi_t\,,\, a_i^+ a_j\; \psi_t \,\rangle_{coh/num} \;\;=\;\; \mathrm{E}\bar{\mathrm{E}}[\, v_i\, \bar{v}_j\,]$$

with the $v_j, \bar{v}_j$ given by the transformed SDE system $\quad (v_j = v_{j,x,y,t},\;\; \bar{v}_j = \bar{v}_{j,x,y,t})$

$$\begin{aligned} dv_j &= -\,i\,(\varepsilon v)_j\,dt \;-\; i\,2u\,p'v_j\bar{v}_j\,v_j\,dt \;-\; i\sqrt{2u}\,v_j\,dx_j \\ d\bar{v}_j &= +\,i\,(\varepsilon \bar{v})_j\,dt \;+\; i\,2u\,p'v_j\bar{v}_j\,\bar{v}_j\,dt \;+\; i\sqrt{2u}\,\bar{v}_j\,dy_j \end{aligned}$$

and

$$p'(v\bar{v}) \;\;=\;\; [\log P]'(v\bar{v}) \;\;=\;\; \begin{cases} 1 & \text{for a coherent state} \\ (N-1)/(v\bar{v}) & \text{for a number state} \end{cases}$$

with $P(x) = e^x$ or x^{N-1} for coh/num .

7. Large N Limit: Gross-Pitaevskii Equation

7.1 GP Equation for the d Dim Bose-Hubbard Model
7.2 GP Equation for the Two Site Bose-Hubbard Model and Numerical Test

7.1 GP Equation for the d Dim Bose-Hubbard Model

7.1 GP Equation for the d Dim Bose-Hubbard Model

Recall $|\lambda|^2 = N$. Define normalized quantities

$$\begin{aligned} w_j &:= v_j \,/\, |\lambda| &&= v_j \,/\, \sqrt{N} \\ \bar{w}_j &:= \bar{v}_j \,/\, |\lambda| &&= \bar{v}_j \,/\, \sqrt{N} \end{aligned}$$

Then, with $p' = p'(v\bar{v}) = p'(Nw\bar{w})$,

$$\begin{aligned} dw_j &= -\, i\, dt\, (\varepsilon w)_j \;-\; i\, 2uN\, dt\, p' w_j \bar{w}_j\, w_j \;-\; i\sqrt{2u}\, w_j\, dx_j \\ d\bar{w}_j &= +\, i\, dt\, (\varepsilon \bar{w})_j \;+\; i\, 2uN\, dt\, p' w_j \bar{w}_j\, \bar{w}_j \;+\; i\sqrt{2u}\, \bar{w}_j\, dy_j \end{aligned}$$

or, with

$$g := uN$$

$$\begin{aligned} dw_j &= -\, i\, dt\, (\varepsilon w)_j \;-\; i\, 2g\, dt\, p' w_j \bar{w}_j\, w_j \;-\; i\sqrt{2g/N}\, w_j\, dx_j \\ d\bar{w}_j &= +\, i\, dt\, (\varepsilon \bar{w})_j \;+\; i\, 2g\, dt\, p' w_j \bar{w}_j\, \bar{w}_j \;+\; i\sqrt{2g/N}\, \bar{w}_j\, dy_j \end{aligned} \tag{3}$$

The large N limit can be read off:

7.1 GP Equation for the d Dim Bose-Hubbard Model

Theorem 5: In the limit $N \to \infty$ with $g = uN$ fixed, Theorem 4 reduces to

$$\langle \psi_t \, , a_i^+ a_j \, \psi_t \rangle_{coh/num} \;\; = \;\; N \;\times\; w_i(t)\, \bar{w}_j(t)$$

with $w, \bar{w}$ given by the ODE system, in both cases coh/num,

$$\begin{aligned} \dot{w}_j &= -\, i\,(\varepsilon w)_j \;-\; i\, 2g\, w_j \bar{w}_j\, w_j \\ \dot{\bar{w}}_j &= +\, i\,(\varepsilon \bar{w})_j \;+\; i\, 2g\, w_j \bar{w}_j\, \bar{w}_j \end{aligned} \tag{4}$$

with initial conditions

$$w_j(0) \;\; = \;\; \lambda_j \,/\, |\lambda| \,, \quad \bar{w}_j(0) \;\; = \;\; \bar{\lambda}_j \,/\, |\lambda|$$

The ODE system (4) is the time dependent discrete Gross-Pitaevskii equation.

The first equation of (4) is enough since $\bar{w}_j$ is the true complex conjugate of w_j now.

7.1 GP Equation for the d Dim Bose-Hubbard Model

Proof: For a coherent state $p' = 1$ and the statement follows immediately from the SDE system (3) since the diffusive part vanishes. For a number state,

$$p' \;=\; p'(Nw\bar{w}) \;=\; (N-1)/(Nw\bar{w}) \;\overset{N\to\infty}{\to}\; 1/(w\bar{w})$$

such that we get the following ODE system:

$$\begin{aligned}
\dot{w}_j &= -\,i\,(\varepsilon w)_j \;-\; i\,2g\,\tfrac{1}{w\bar{w}}\,w_j\bar{w}_j\,w_j \\
\dot{\bar{w}}_j &= +\,i\,(\varepsilon \bar{w})_j \;+\; i\,2g\,\tfrac{1}{w\bar{w}}\,w_j\bar{w}_j\,\bar{w}_j
\end{aligned}$$

However,

$$\begin{aligned}
\tfrac{d}{dt}(w\bar{w}) &= \textstyle\sum_j \Big\{ -\,i\,(\varepsilon w)_j\bar{w}_j \;-\; i\,2g\,\tfrac{1}{w\bar{w}}\,(w_j\bar{w}_j)^2 \;+\; i\,w_j(\varepsilon\bar{w})_j \;+\; i\,2g\,\tfrac{1}{w\bar{w}}\,(w_j\bar{w}_j)^2 \Big\} \\
&= -\,i\,(\varepsilon w)\cdot\bar{w} \;+\; i\,w\cdot(\varepsilon\bar{w}) \;\overset{\varepsilon=\varepsilon^T}{=}\; 0
\end{aligned}$$

which results in $(w\bar{w})_t = (w\bar{w})_0 = 1$. ■

7.2 GP Equation for the Two Site Bose-Hubbard Model and Numerical Test

7.2 GP Equation for the Two Site Bose-Hubbard Model

For just two lattice sites 1 and 2, we get from Theorem 5

$$\langle n_{j,t} \rangle \;:=\; \langle\, \psi_t \,,\, a_j^+ a_j \,\psi_t \,\rangle_{coh/num} \;\overset{N\to\infty}{=}\; N\,|w_{j,t}|^2 \;=:\; N\,\varrho_{j,t}$$

with w_1, w_2 given by the GP system

$$\begin{aligned} \dot{w}_1 &= -\,i\,\varepsilon\,w_2 \;-\; i\,2g\,|w_1|^2\,w_1 \\ \dot{w}_2 &= -\,i\,\varepsilon\,w_1 \;-\; i\,2g\,|w_2|^2\,w_2 \;. \end{aligned}$$

Theorem 6 (well known)**:** Introduce the normalized particle imbalance and its integral,

$$\begin{aligned} \varrho_{12} &:= |w_1|^2 \;-\; |w_2|^2 \\ \varphi_t &:= 2g\int_0^t \varrho_{12,s}\,ds \end{aligned}$$

Then φ_t is a solution of

$$\ddot{\varphi}_t \;+\; 4\varepsilon^2 \sin\varphi_t \;=\; 0$$

and the density of particles at lattice site 1 is obtained as

$$\varrho_{1,t} \;=\; |w_{1,t}|^2 \;=\; \tfrac{1}{2}\big(1 \,+\, \tfrac{\dot{\varphi}_t}{2g}\big) \;.$$

7.2 GP Equation for the Two Site Bose-Hubbard Model

Initial conditions: All particles at lattice site 1 at $t = 0$. Then $\varphi_0 = 0$, always, and

$$\dot{\varphi}_0 \;=\; 2g\,\varrho_{12,0} \;=\; 2g\,(\varrho_{1,0} - \varrho_{2,0}) \;=\; 2g\,(1-0) \;=\; 2g$$

Total energy for mathematical pendulum ODE $\ddot{\varphi}_t + 4\varepsilon^2 \sin\varphi_t = 0$ is

$$E \;=\; \frac{\dot{\varphi}_t^2}{2} - 4\varepsilon^2 \cos\varphi_t \;=\; \frac{\dot{\varphi}_0^2}{2} - 4\varepsilon^2 \cos\varphi_0 \;=\; 2g^2 - 4\varepsilon^2$$

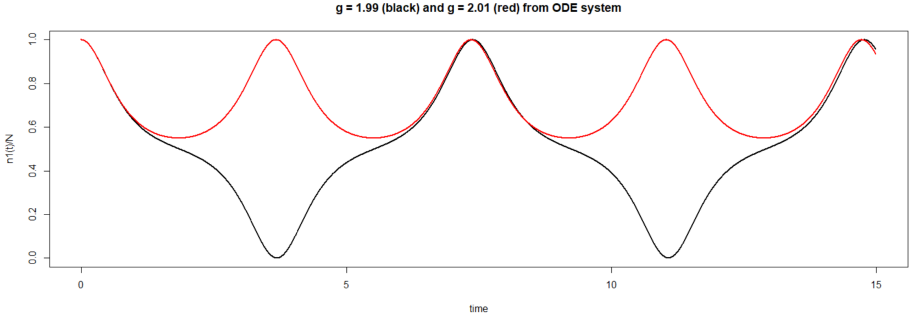
Potential energy at $\varphi = \pi$ is $E_{pot} = +4\varepsilon^2$. We have **rollovers** if the total energy is bigger than that:

$$\begin{aligned} & 2g^2 - 4\varepsilon^2 \;>\; +4\varepsilon^2 \\ \Leftrightarrow \qquad & \boldsymbol{g^2 \;>\; (2\varepsilon)^2} \end{aligned}$$

In that case, the velocity $\dot{\varphi}_t = 2g\,(\varrho_{1,t} - \varrho_{2,t})$ which is the particle imbalance between the two lattice sites, does not change its sign and this corresponds to the non-oscillatory or self-trapping regime:

7.2 GP Equation for the Two Site Bose-Hubbard Model

The mathematical pendulum ODE for $\varepsilon = 1$ and $g \in \{1.99, 2.01\}$:



the quantity $\varrho_{1,t} = \frac{1}{2}\left(1 + \frac{\dot{\varphi}_t}{2g}\right)$ with φ_t from ODE, $\varphi_0 = 0$ and $\dot{\varphi}_0 = 2g$

7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test

Numerical check of the large N limit: $\varepsilon = 1$ and

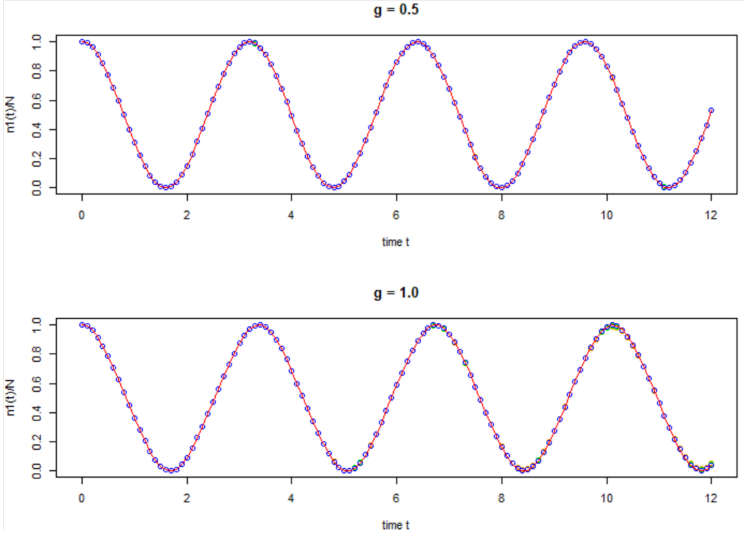
$$\begin{aligned} N &\in \{\, 2500\,,\, 5000\,,\, 10000\,,\, 20000 \,\} \;=\; \{\,\text{orange}\,,\,\text{green}\,,\,\text{light blue}\,,\,\text{dark blue}\,\} \\ g &\in \{\, 0.5\,,\, 1.0\,,\, 1.8\,,\, 2.2\,,\, 3.0\,,\, 6.0 \,\} \end{aligned}$$

Quantity:

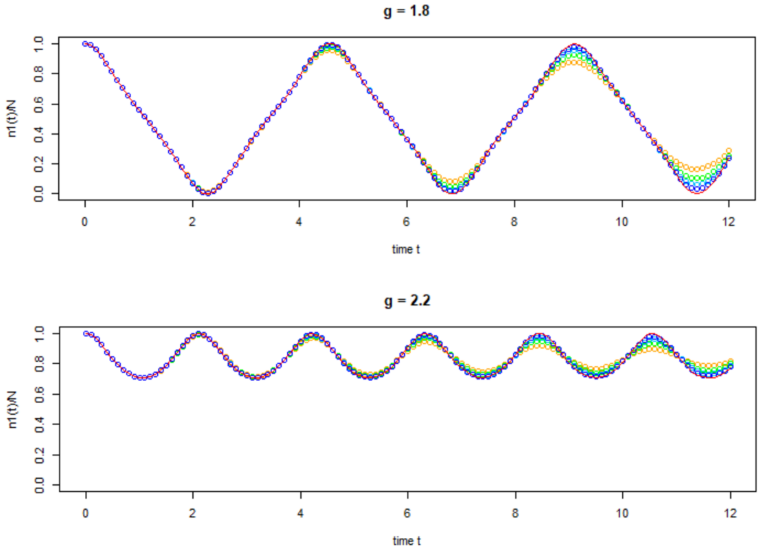
$$\varrho_{1,t} \;=\; \begin{cases} \langle \psi_t, a_1^+ a_1\, \psi_t \rangle \;/\; N & \text{from exact diagonalization} \\ \frac{1}{2}\big(1 + \frac{\dot{\varphi}_t}{2g}\big) & \text{from ODE mathematical pendulum} \end{cases}$$

The red line below is the ODE solution and the dots come from exact diagonalization:

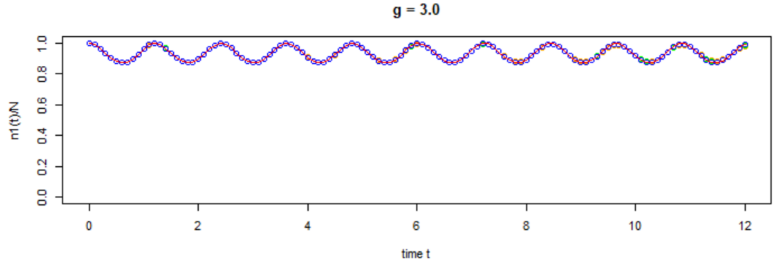
7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test

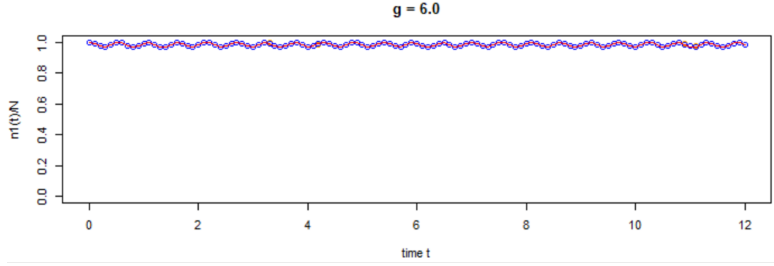


7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test



7.2 GP Equation for the Two Site Bose-Hubbard Model: Numerical Test





8. Summary

8. Summary

Main Results:

- Simple and Elegant Method to Calculate GP-like Mean Field Equations
- Generic Formalism: Arbitrary Initial State, Arbitrary Hopping Matrix, Arbitrary Dimension
- Various Exact PDE Representations
- collapse and revivals could be reproduced, proper treatment of diffusive part still missing

Outlook:

- Use this for New Numerical or Analytical Calculation Schemes
- Fermi Hubbard Model
- Thermodynamic Quantities

full paper at

`https://arxiv.org/abs/2205.02010`

8. Summary

Theorems:

Theorem 1: Time Evolution of States as Fresnel Expectation Values, Matrix U_{kdt}

Theorem 2: Correlation Functions as Fresnel Expectation Values

Theorem 3: SDE for U_{kdt} and Correlation Functions as Expectations of Fresnel Diffusions

Theorem 4: Girsanov Transformed SDE System: Initial State moves into Drift Part

Theorem 5: Large N-Limit of Theorem 4: Gross-Pitaevskii Equation

Theorem 6: GP Equation for Two Site Bose-Hubbard Model as Mathematical Pendulum

Appendix:

Theorem 7: PDE Representation Correlation Functions, d-dim Bose-Hubbard Model

Theorem 8: PDE Representation Correlation Functions, Two Site Bose-Hubbard Model

Theorem 9: Equivalence Mathematical Pendulum and Quartic Double-Well Potential

Standard Formula Wiener and Fresnel Expectations

Appendix

- PDE Representation d Dim Bose-Hubbard Model
- PDE Representation Two Site Bose-Hubbard Model
- Equivalence Mathematical Pendulum and Quartic Double-Well Potential
- Standard Formula Wiener and Fresnel Expectations

Appendix: PDE Representation d Dim Bose-Hubbard Model

Theorem 7: Define the differential operators

$$\mathcal{L}_0 \;:=\; \sum_{i,j} \varepsilon_{ij} \left\{ v_i \frac{\partial}{\partial v_j} - \bar{v}_i \frac{\partial}{\partial \bar{v}_j} \right\}, \quad \mathcal{L}_{int} \;:=\; u \sum_j \left\{ v_j^2 \frac{\partial^2}{\partial v_j^2} - \bar{v}_j^2 \frac{\partial^2}{\partial \bar{v}_j^2} \right\}$$

and for

$$P(x) \;:=\; \begin{cases} e^x & \text{for coherent state} \\ x^{N-1} & \text{for number state} \end{cases}$$

let

$$\mathcal{L}_P \;:=\; 2u \frac{P'(v\bar{v})}{P(v\bar{v})} \sum_j v_j \bar{v}_j \left\{ v_j \frac{\partial}{\partial v_j} - \bar{v}_j \frac{\partial}{\partial \bar{v}_j} \right\}$$

Then, for the d-dimensional Bose-Hubbard model with hopping matrix ε_{ij} and interaction u,

$$\begin{aligned} \langle \psi_t \,,\, a_i^+ a_j \; \psi_t \rangle_{coh/num} &= e^{-it(\mathcal{L}_0 + \mathcal{L}_{int})} \left\{ v_i \, \bar{v}_j \; P(v\bar{v}) \right\} / \, P(\lambda\bar{\lambda}) \;\; |_{v=\lambda, \bar{v}=\bar{\lambda}} \\ &= e^{-it(\mathcal{L}_0 + \mathcal{L}_{int} + \mathcal{L}_P)} \left\{ v_i \, \bar{v}_j \right\} \;\; |_{v=\lambda, \bar{v}=\bar{\lambda}} \;\; . \end{aligned}$$

Appendix: PDE Representation Two Site Bose-Hubbard Model

Theorem 8: Define the differential operators (with $p' = [\log P]'$, $P(x) = e^x$ or x^{N-1} coh/num, $n = n_1 + n_2$)

$$\mathcal{L}_\varepsilon := -\varepsilon \Big\{ (q-\bar{q}) \big(\tfrac{\partial}{\partial n_1} - \tfrac{\partial}{\partial n_2} \big) + (n_1 - n_2) \big(\tfrac{\partial}{\partial q} - \tfrac{\partial}{\partial \bar{q}} \big) \Big\}$$

$$\mathcal{L}_u := +2u \Big\{ \big(n_1 \tfrac{\partial}{\partial n_1} - n_2 \tfrac{\partial}{\partial n_2} \big) \big(q \tfrac{\partial}{\partial q} - \bar{q} \tfrac{\partial}{\partial \bar{q}} \big) + p'(n)(n_1 - n_2) \big(q \tfrac{\partial}{\partial q} - \bar{q} \tfrac{\partial}{\partial \bar{q}} \big) \Big\}$$

acting on functions of 4 variables $F = F(n_1, n_2, q, \bar{q})$. Then for the two site Bose-Hubbard model

$$\langle \psi_t, a_1^+ a_1 \, \psi_t \rangle_{coh/num} = e^{-it(\mathcal{L}_\varepsilon + \mathcal{L}_u)} \, n_1 \Big|_{(n_1, n_2, q, \bar{q}) = (|\lambda_1|^2, |\lambda_2|^2, \lambda_1 \bar{\lambda}_2, \bar{\lambda}_1 \lambda_2)}$$

with actions

$$(e^{-it\mathcal{L}_\varepsilon} F)(n_1, n_2, q, \bar{q}) = F\big(R_t \, (n_1, n_2, q, \bar{q})^T \big)$$

$$R_t = \begin{pmatrix} \cos^2 \varepsilon t & \sin^2 \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t \\ \sin^2 \varepsilon t & \cos^2 \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t \\ +i \sin \varepsilon t \cos \varepsilon t & -i \sin \varepsilon t \cos \varepsilon t & \cos^2 \varepsilon t & \sin^2 \varepsilon t \\ -i \sin \varepsilon t \cos \varepsilon t & +i \sin \varepsilon t \cos \varepsilon t & \sin^2 \varepsilon t & \cos^2 \varepsilon t \end{pmatrix}$$

Appendix: PDE Representation Two Site Bose Hubbard Model

and

$$e^{-it\mathcal{L}_u}\left\{\, G(n_1,n_2)\, q^b\, \bar{q}^{\,\bar{b}}\,\right\} \;=\; G\big(\, e^{-i2ut\,(b-\bar{b})}n_1\,,\; e^{+i2ut\,(b-\bar{b})}n_2\,\big)\;\times\;\frac{P\big(\, e^{-i2ut\,(b-\bar{b})}n_1\;+\;e^{+i2ut\,(b-\bar{b})}n_2\,\big)}{P(n_1+n_2)}\;\times\; q^b\,\bar{q}^{\,\bar{b}}$$

where $G = G(n_1, n_2)$ is an arbitrary analytic function and $b, \bar{b}$ are arbitrary natural numbers.

Example, used for Collapse and Revivals:

$$\begin{aligned}
e^{-it\mathcal{L}_u}\left\{\, n_1\,\right\} &= n_1\\
e^{-it\mathcal{L}_u}\left\{\, q\,\right\} &= \frac{P\big(\, e^{-i2ut}n_1\;+\;e^{+i2ut}n_2\,\big)}{P(n_1+n_2)}\;\times\; q\\
e^{-it\mathcal{L}_u}\left\{\, n_1 q\,\right\} &= e^{-i2ut}\,n_1\;\times\;\frac{P\big(\, e^{-i2ut}n_1\;+\;e^{+i2ut}n_2\,\big)}{P(n_1+n_2)}\;\times\; q
\end{aligned}$$

such that

$$\frac{e^{-it\mathcal{L}_u}[n_1 q]}{e^{-it\mathcal{L}_u}[n_1]\,e^{-it\mathcal{L}_u}[q]} \;=\; e^{-i2ut}\,,\qquad \frac{e^{-it\mathcal{L}_\varepsilon}[n_1 q]}{e^{-it\mathcal{L}_\varepsilon}[n_1]\,e^{-it\mathcal{L}_\varepsilon}[q]} \;=\; 1\;.$$

Theorem 9: The mathematical pendulum

$$\ddot{\varphi}_t \;+\; 4\varepsilon^2 \sin\varphi_t \;\;=\;\; 0$$

with $\varphi_0 = 0$ and $\dot{\varphi}_0 = 2g$ is equivalent to

$$\ddot{x}_t \;+\; (4\varepsilon^2 - 2g^2)\, x_t \;+\; 2g^2\, x_t^3 \;\;=\;\; 0$$

with $x_0 = 1$ and $\dot{x}_0 = 0$ through the transformation

$$\varphi_t \;\;=\;\; 2g \int_0^t x_s \, ds \qquad \Leftrightarrow \qquad x_t \;\;=\;\; \tfrac{1}{2g}\, \dot{\varphi}_t \;\; .$$

Appendix: Standard Formula Wiener and Fresnel Expectations

Theorem: Consider m times $0 < t_1 < t_2 < \cdots < t_m \leq T$ and let x_{t_j} be a standard or Fresnel Brownian motion observed at time t_j. Let

$$F \;=\; F(x_{t_1}, \cdots, x_{t_m}) \;:\; \mathbb{R}^m \;\to\; \mathbb{C}$$

be an arbitrary function of m variables and let $\mathrm{E}[F]$ denote its Wiener or Fresnel expectation value. Then, with $t_0 := 0$ and $x_0 := 0$,

$$\mathrm{E}[\,F\,] \;=\; \int_{\mathbb{R}^m} F(x_{t_1}, \cdots, x_{t_m}) \, \prod_{j=1}^{m} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}) \, dx_{t_j}$$

with Gaussian or Fresnel kernels given by

$$p_t(x,y) \;:=\; \begin{cases} \frac{1}{\sqrt{2\pi t}} \; e^{-\frac{(x-y)^2}{2t}} & \text{for Wiener expectations} \\ \frac{1}{\sqrt{2\pi i t}} \; e^{\,i\,\frac{(x-y)^2}{2t}} & \text{for Fresnel expectations}\,. \end{cases}$$

Basic property: $\int_{\mathbb{R}} p_t(x,y)\, p_s(y,z)\, dy \;=\; p_{t+s}(x,z)$.