Hochschule RheinMain
Prof. Dr. D. Lehmann WS 2023/24

Lösungen zum 11. Übungsblatt
Dynamik der Teilchen und Felder

2.Aufgabe: Mit der mehrdimensionalen Kettenregel bekommen wir

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial \varepsilon} &= \frac{\partial H}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi} &= \frac{\partial H}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \varphi}\end{aligned}$$

oder, wenn wir die Hamiltonschen Gleichungen

$$\dot{x} = \frac{\partial H}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

benutzen,

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial \varepsilon} &= -\frac{\partial x}{\partial \varepsilon}\,\dot{p} + \frac{\partial p}{\partial \varepsilon}\,\dot{x} \\ \frac{\partial \tilde{H}}{\partial \varphi} &= -\frac{\partial x}{\partial \varphi}\,\dot{p} + \frac{\partial p}{\partial \varphi}\,\dot{x}\end{aligned}$$

In Matrix-Notation,

$$\begin{pmatrix}\frac{\partial \tilde{H}}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi}\end{pmatrix} = \begin{pmatrix}\frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{p}\end{pmatrix}$$

Weiterhin,

$$\begin{aligned}\dot{x} &= \frac{\partial x}{\partial \varepsilon}\,\dot{\varepsilon} + \frac{\partial x}{\partial \varphi}\,\dot{\varphi} \\ \dot{p} &= \frac{\partial p}{\partial \varepsilon}\,\dot{\varepsilon} + \frac{\partial p}{\partial \varphi}\,\dot{\varphi}\end{aligned}$$

oder in Matrix-Notation,

$$\begin{pmatrix}\dot{x} \\ \dot{p}\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\dot{\varepsilon} \\ \dot{\varphi}\end{pmatrix}$$

Also,

$$\begin{pmatrix}\frac{\partial \tilde{H}}{\partial \varepsilon} \\ \frac{\partial \tilde{H}}{\partial \varphi}\end{pmatrix} = \begin{pmatrix}\frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\dot{\varepsilon} \\ \dot{\varphi}\end{pmatrix} =: J\begin{pmatrix}\dot{\varepsilon} \\ \dot{\varphi}\end{pmatrix} \tag{1}$$

mir der Matrix

$$\begin{aligned}J &:= \begin{pmatrix}\frac{\partial p}{\partial \varepsilon} & -\frac{\partial x}{\partial \varepsilon} \\ \frac{\partial p}{\partial \varphi} & -\frac{\partial x}{\partial \varphi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial \varepsilon} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial p}{\partial \varepsilon} & \frac{\partial p}{\partial \varphi}\end{pmatrix} \\ &= \begin{pmatrix}0 & \frac{\partial p}{\partial \varepsilon}\frac{\partial x}{\partial \varphi} - \frac{\partial x}{\partial \varepsilon}\frac{\partial p}{\partial \varphi} \\ \frac{\partial p}{\partial \varphi}\frac{\partial x}{\partial \varepsilon} - \frac{\partial x}{\partial \varphi}\frac{\partial p}{\partial \varepsilon} & 0\end{pmatrix} =: \begin{pmatrix}0 & -j(\varepsilon,\varphi) \\ j(\varepsilon,\varphi) & 0\end{pmatrix}\end{aligned} \tag{2}$$

mit der skalaren Funktion

$$j(\varepsilon, \varphi) \quad := \quad \frac{\partial x}{\partial \varepsilon} \frac{\partial p}{\partial \varphi} \; - \; \frac{\partial x}{\partial \varphi} \frac{\partial p}{\partial \varepsilon}$$

Wir haben

$$\begin{aligned}
\frac{\partial x}{\partial \varepsilon} \quad &= \quad \frac{1}{\sqrt{2\varepsilon}} \cos \varphi \\
\frac{\partial x}{\partial \varphi} \quad &= \quad -\sqrt{2\varepsilon} \sin \varphi \\
\frac{\partial p}{\partial \varepsilon} \quad &= \quad \frac{1}{\sqrt{2\varepsilon}} \sin \varphi \\
\frac{\partial p}{\partial \varphi} \quad &= \quad \sqrt{2\varepsilon} \cos \varphi
\end{aligned}$$

und bekommen somit

$$j(\varepsilon, \varphi) \quad = \quad \frac{\partial x}{\partial \varepsilon} \frac{\partial p}{\partial \varphi} \; - \; \frac{\partial x}{\partial \varphi} \frac{\partial p}{\partial \varepsilon} \quad = \quad \cos^2 \varphi \, + \, \sin^2 \varphi \quad = \quad 1 \; .$$

Setzt man das in die Gleichungen (2) und (1) ein, folgt die Behauptung.