Hochschule RheinMain
Prof. Dr. D. Lehmann

WS 2022/23

Lösungen 9. Übungsblatt Quantenmechanik

Aufgabe 1: a) Wir haben

$$
\begin{aligned}
-L_1^2 &= \big(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} \big)^2 \\
&= x_2^2 \frac{\partial^2}{\partial x_3^2} + x_3^2 \frac{\partial^2}{\partial x_2^2} - x_2 \frac{\partial}{\partial x_3} x_3 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_2} x_2 \frac{\partial}{\partial x_3} \\
&= x_2^2 \frac{\partial^2}{\partial x_3^2} + x_3^2 \frac{\partial^2}{\partial x_2^2} - 2 x_2 x_3 \frac{\partial^2}{\partial x_2 \partial x_3} - x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3}
\end{aligned}
$$

und analoge Formeln für L_2^2 und L_3^2 durch zyklische Vertauschung der Indizes. Damit bekommen wir

$$
\begin{aligned}
-\vec{L}^{\,2} &= (x_1^2 + x_2^2) \frac{\partial^2}{\partial x_3^2} + (x_2^2 + x_3^2) \frac{\partial^2}{\partial x_1^2} + (x_3^2 + x_1^2) \frac{\partial^2}{\partial x_2^2} \\
&\quad - 2 \big\{ x_2 x_3 \frac{\partial^2}{\partial x_2 \partial x_3} + x_3 x_1 \frac{\partial^2}{\partial x_3 \partial x_1} - x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} \big\} - 2 \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} \\
&= r^2 \Delta - \sum_{i=1}^{3} x_i^2 \frac{\partial^2}{\partial x_i^2} - 2 \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} - 2 \big\{ x_2 x_3 \frac{\partial^2}{\partial x_2 \partial x_3} + \text{zyklisch} \big\} \; .
\end{aligned}
$$

b) Mit der Kettenregel bekommen wir

$$
\frac{\partial}{\partial r} = \frac{\partial x_1}{\partial r} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial r} \frac{\partial}{\partial x_2} + \frac{\partial x_3}{\partial r} \frac{\partial}{\partial x_3}
$$

In Kugelkoordinaten

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = r \begin{pmatrix} \cos\varphi \, \sin\theta \\ \sin\varphi \, \sin\theta \\ \cos\theta \end{pmatrix}
$$

gilt offensichtlich

$$
\frac{\partial x_i}{\partial r} = \frac{x_i}{r}
$$

Also

$$
\frac{\partial}{\partial r} = \frac{x_1}{r} \frac{\partial}{\partial x_1} + \frac{x_2}{r} \frac{\partial}{\partial x_2} + \frac{x_3}{r} \frac{\partial}{\partial x_3}
$$

und damit

$$r\frac{\partial}{\partial r} \;=\; \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j}$$

Weiterhin

$$\left(r\frac{\partial}{\partial r}\right)^2 \;=\; r\frac{\partial}{\partial r}\, r\frac{\partial}{\partial r} \;=\; r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} \;\; .$$

c) Aus

$$\Delta \;=\; \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S$$

folgt

$$\Delta_S \;=\; r^2\Delta - r^2\frac{\partial^2}{\partial r^2} - 2r\frac{\partial}{\partial r} \;=\; r^2\Delta - \left(r\frac{\partial}{\partial r}\right)^2 - r\frac{\partial}{\partial r}$$

Nun ist

$$\begin{aligned}
\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r} \;&=\; \sum_{i,j=1} x_i\frac{\partial}{\partial x_i}\, x_j\frac{\partial}{\partial x_j} + \sum_{i=1} x_i\frac{\partial}{\partial x_i} \\
&=\; \sum_{\substack{i,j=1\\ i\neq j}} x_i\frac{\partial}{\partial x_i}\, x_j\frac{\partial}{\partial x_j} + \sum_{i=1} x_i\frac{\partial}{\partial x_i}\, x_i\frac{\partial}{\partial x_i} + \sum_{i=1} x_i\frac{\partial}{\partial x_i} \\
&=\; \sum_{\substack{i,j=1\\ i\neq j}} x_i x_j\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1} x_i^2\frac{\partial^2}{\partial x_i^2} + 2\sum_{i=1} x_i\frac{\partial}{\partial x_i} \\
&=\; 2\left\{ x_2x_3\frac{\partial^2}{\partial x_2\partial x_3} + \text{zyklisch}\right\} + \sum_{i=1} x_i^2\frac{\partial^2}{\partial x_i^2} + 2\sum_{i=1} x_i\frac{\partial}{\partial x_i}
\end{aligned}$$

Mit Teil (a) bekommen wir dann

$$\begin{aligned}
-\vec{L}^2 \;&=\; r^2\Delta - \sum_{i=1}^{3} x_i^2\frac{\partial^2}{\partial x_i^2} - 2\sum_{i=1}^{3} x_i\frac{\partial}{\partial x_i} - 2\left\{ x_2x_3\frac{\partial^2}{\partial x_2\partial x_3} + \text{zyklisch}\right\} \\
&=\; r^2\Delta - \left(r\frac{\partial}{\partial r}\right)^2 - r\frac{\partial}{\partial r} \;=\; \Delta_S \;\; .
\end{aligned}$$

Aufgabe 2: Wir bekommen

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^m \left\{ (1-x^2)\,P'' - 2x\,P' + \lambda\,P \right\} \\
&\quad=\; \binom{m}{2}(-2)\,P^{(m)}(x) + \binom{m}{1}(-2x)\,P^{(m+1)}(x) + \binom{m}{0}(1-x^2)\,P^{(m+2)}(x) \\
&\qquad + \binom{m}{1}(-2)\,P^{(m)}(x) + \binom{m}{0}(-2x)\,P^{(m+1)}(x) + \lambda\,P^{(m)}(x) \\
&\quad=\; (1-x^2)\,P^{(m+2)}(x) - 2(m+1)x\;P^{(m+1)}(x) + [-m(m-1) - 2m + \lambda]\,P^{(m)}(x)
\end{aligned}$$

oder

$$\left(\frac{d}{dx}\right)^m \Big\{ (1-x^2)\,P'' \;-\; 2x\,P' \;+\; \lambda\,P \Big\} \;=$$

$$(1-x^2)\,P^{(m+2)}(x) \;-\; 2(m+1)x\;P^{(m+1)}(x) \;+\; [\lambda-(m^2+m)]\,P^{(m)}(x) \qquad (1)$$

Nun ist

$$\begin{aligned}
(1-x^2)\,p''(x) \;-\; 2x\,p'(x) \;&=\; \frac{d}{dx}(1-x^2)\frac{d}{dx}\,p \\
&=\; \frac{d}{dx}(1-x^2)\frac{d}{dx}\Big\{ (1-x^2)^{\frac{m}{2}}\,P^{(m)}(x) \Big\} \\
&=\; \frac{d}{dx}(1-x^2)\Big\{ \frac{m}{2}(-2x)(1-x^2)^{\frac{m}{2}-1}\,P^{(m)}(x) \;+(1-x^2)^{\frac{m}{2}}\,P^{(m+1)}(x) \Big\} \\
&=\; \frac{d}{dx}\Big\{ (-mx)(1-x^2)^{\frac{m}{2}}\,P^{(m)}(x) \;+\; (1-x^2)^{\frac{m}{2}+1}\,P^{(m+1)}(x) \Big\} \\
&=\; (-m)(1-x^2)^{\frac{m}{2}}\,P^{(m)} \;+\; (-mx)^2(1-x^2)^{\frac{m}{2}-1}\,P^{(m)} \;+\; (-mx)(1-x^2)^{\frac{m}{2}}\,P^{(m+1)} \\
&\qquad +\big(\tfrac{m}{2}+1\big)(-2x)(1-x^2)^{\frac{m}{2}}\,P^{(m+1)} \;+\; (1-x^2)^{\frac{m}{2}+1}\,P^{(m+2)} \\
&=\; (1-x^2)^{\frac{m}{2}}\,\Big\{ -m\,P^{(m)} \;+\; \frac{m^2x^2}{1-x^2}\,P^{(m)} \;-\; mx\,P^{(m+1)} \\
&\qquad\qquad +\; \big(\tfrac{m}{2}+1\big)(-2x)\,P^{(m+1)} \;+\; (1-x^2)\,P^{(m+2)} \Big\}
\end{aligned}$$

oder

$$(1-x^2)\,p''(x) \;-\; 2x\,p'(x) \;=$$

$$(1-x^2)^{\frac{m}{2}}\,\Big\{(1-x^2)\,P^{(m+2)} \;-\; (m+1)2x\,P^{(m+1)} \;+\; \Big[\frac{m^2}{1-x^2} \;-\; (m+m^2)\Big]\,P^{(m)}\Big\}$$

Damit ergibt sich dann

$$\begin{aligned}
(1-x^2)\,p''(x) \;-\; 2x\,p'(x) \;+\; \big[\,\lambda \;-\; \frac{m^2}{1-x^2}\,\big]\,p \;&= \\
&(1-x^2)^{\frac{m}{2}}\,\Big\{(1-x^2)\,P^{(m+2)} \;-\; (m+1)2x\,P^{(m+1)} \;+\; \Big[\frac{m^2}{1-x^2} \;-\; (m+m^2)\Big]\,P^{(m)}\Big\} \\
&+\big[\,\lambda \;-\; \frac{m^2}{1-x^2}\,\big]\,(1-x^2)^{\frac{m}{2}}\,P^{(m)} \\
=\;\; &(1-x^2)^{\frac{m}{2}}\,\Big\{(1-x^2)\,P^{(m+2)} \;-\; (m+1)2x\,P^{(m+1)} \;+\; \Big[\lambda \;-\; (m+m^2)\Big]\,P^{(m)}\Big\} \\
\overset{(1)}{=}\;\; &(1-x^2)^{\frac{m}{2}}\,\left(\frac{d}{dx}\right)^m \Big\{ (1-x^2)\,P'' \;-\; 2x\,P' \;+\; \lambda\,P \Big\}\;. \qquad \blacksquare
\end{aligned}$$