week4b: Kapitel 3: Real World and Risk Neutral Probabilities, Teil2

Letztes Mal hatten wir für den Fall Zinsen $r = 0$ die folgende Formel für den Preis V_0 einer pfadabhängigen oder pfadunabhängigen Option H hergeleitet:

$$V_0 \;=\; \mathsf{E}_{\rm rn}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{1}$$

Dabei war die risikoneutrale W'keit $p_{\rm rn}$ gegeben durch

$$p_{\rm rn} \;=\; \frac{-{\rm ret}_{\rm down}}{{\rm ret}_{\rm up} - {\rm ret}_{\rm down}} \;=:\; p_{\text{risk neutral}} \tag{2}$$

und wir hatten das Binomialmodell stochastisch gemacht, indem wir gesagt hatten, up-Returns sollen mit W'keit $p = p_{\rm rn}$ eintreten und down-Returns mit W'keit $1 - p_{\rm rn}$. Die Formel für das $p_{\rm rn}$ hatten wir durch die Forderung

$$\mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;\stackrel{!}{=}\; S_k \tag{3}$$

bekommen, und wegen dem Bestehen dieser Gleichung hatten wir dann die Identität

$$\begin{aligned} &\mathsf{E}_{\rm rn}\Big[\, \delta_{k-1}(S_0, ..., S_{k-1}) \times (S_k - S_{k-1}) \,\Big|\, S_0 \,\Big] \\ &= \; \mathsf{E}_{\rm rn}\Big[\, \delta_{k-1}(S_0, ..., S_{k-1}) \times \mathsf{E}_{\rm rn}\Big[\, S_k - S_{k-1} \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \,\Big|\, S_0 \,\Big] \\ &= \; \mathsf{E}_{\rm rn}\Big[\, \delta_{k-1}(S_0, ..., S_{k-1}) \times \underbrace{\Big(\mathsf{E}_{\rm rn}\Big[\, S_k \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \;-\; S_{k-1}\Big)}_{=0} \,\Big|\, S_0 \,\Big] \;=\; 0 \end{aligned} \tag{4}$$

erhalten und daraus folgte dann sofort die Pricing-Formel (1). Der Fall mit Zinsen $r \neq 0$ geht jetzt ziemlich analog:

Theorem 3.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model with ${\rm ret}_k \in \{{\rm ret}_{\rm up}, {\rm ret}_{\rm down}\}$. Let r be the interest rate paid per period and $R := 1 + r$. Let

$$s_k \;=\; R^{-k}\, S_k \tag{5}$$

denote the discounted price process. Then the following statements hold:

a) Define the risk neutral probability

$$p_{\rm rn} \;=\; p_{\text{risk neutral}} \;:=\; \frac{r \;-\; {\rm ret}_{\rm down}}{{\rm ret}_{\rm up} \;-\; {\rm ret}_{\rm down}} \tag{6}$$

and denote expectations with respect to this probability by $\mathsf{E}_{\rm rn}[\,\cdot\,]$. Then the discounted price process $\{s_k\}_{k=0}^N$ is a martingale with respect to the risk neutral expectation. That is, the following equation holds for all $k = 0, 1, 2, ..., N-1$:

$$\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{s_j\}_{j=0}^k \,\big] \;=\; s_k \tag{7}$$

b) Let $H = H(S_0, S_1, ..., S_N)$ be the payoff of some option. Then the theoretical fair value of this option can be obtained from the following risk neutral expectation:

$$V_0 \;=\; R^{-N}\, \mathsf{E}_{\rm rn}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{8}$$

Proof: In the presence of non zero interest rates the zero rates equation

$$H(S_0, S_1, ..., S_N) \;=\; V_0 \;+\; \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (S_k - S_{k-1})$$

is substituted by

$$h(S_0, S_1, ..., S_N) \;=\; v_0 + \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \tag{9}$$

with $h = R^{-N} H$ being the discounted payoff function and $s_k = R^{-k} S_k$ being the discounted underlying prices. Thus, if we want to eliminate the sum on the right hand side of (9) by taking an expectation value, we need to have the following property:

$$\mathsf{E}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] \;\stackrel{!}{=}\; s_k \tag{10}$$

or

$$R^{-(k+1)}\, \mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] \;\stackrel{!}{=}\; R^{-k}\, S_k \tag{11}$$

which is equivalent to

$$\begin{aligned} S_k \times \big(1 \,+\, \mathsf{E}\big[\,\mathrm{ret}_{k+1} \,|\, \{S_j\}_{j=0}^{k}\,\big]\,\big) \;&\stackrel{!}{=}\; R\, S_k \\ S_k \times \Big(1 \,+\, \mathrm{ret}_{\rm up} \cdot p \,+\, \mathrm{ret}_{\rm down} \cdot (1-p)\Big) \;&\stackrel{!}{=}\; R\, S_k \\ \Leftrightarrow \quad \mathrm{ret}_{\rm up} \cdot p \,+\, \mathrm{ret}_{\rm down} \cdot (1-p) \;&=\; R - 1 \\ \Leftrightarrow \quad (\mathrm{ret}_{\rm up} - \mathrm{ret}_{\rm down}) p \;&=\; r - \mathrm{ret}_{\rm down} \end{aligned}$$

which gives

$$p \;=\; \frac{r - \mathrm{ret}_{\rm down}}{\mathrm{ret}_{\rm up} - \mathrm{ret}_{\rm down}} \;=:\; p_{\text{risk neutral}} \;=:\; p_{\rm rn}$$

This proves part (a). Part (b) is obtained in the same way as (1) is obtained through (3) and (4):

$$\begin{aligned} \mathsf{E}_{\rm rn}\big[\, h(S_0, S_1, ..., S_N)\,\big] \;&=\; v_0 + \sum_{k=1}^{N} \mathsf{E}_{\rm rn}\Big[\; \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \;\Big] \\ &=\; v_0 \,+\, 0 \;=\; v_0 \;=\; V_0 \end{aligned}$$

since

$$\begin{aligned} &\mathsf{E}_{\rm rn}\Big[\, \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \,\Big|\, S_0 \Big] \\ &=\; \mathsf{E}_{\rm rn}\Big[\; \delta_{k-1}(S_0, ..., S_{k-1}) \times \mathsf{E}_{\rm rn}\Big[\, s_k - s_{k-1} \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \;\Big|\; S_0 \Big] \\ &=\; \mathsf{E}_{\rm rn}\Big[\; \delta_{k-1}(S_0, ..., S_{k-1}) \times \underbrace{\Big(\mathsf{E}_{\rm rn}\Big[\, s_k \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \;-\; s_{k-1}\Big)}_{=\,0 \;\text{ because of (10)}} \;\Big|\; S_0 \Big] \;=\; 0 \qquad \blacksquare \end{aligned}$$

Let us also note the following

Corollary 3.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model with $\text{ret}_k \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\}$. Let r be the interest rates per period and let

$$v_k \;=\; R^{-k}\, V_k \tag{12}$$

denote the discounted portfolio value of the replicating portfolio V_k. Then the process $\{v_k\}_{k=0}^{N}$ is a martingale with respect to expectations with the risk neutral probability (6). That is,

$$\mathsf{E}_{\text{rn}}\big[\, v_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] \;=\; v_k \tag{13}$$

for all $k = 0, 1, 2, ..., N-1$.

Proof: According to part (b) of Theorem 1.1 we have

$$v_k \;=\; v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1})$$

from which we get

$$v_{k+1} \;=\; v_k + \delta_k(s_{k+1} - s_k) \tag{14}$$

Thus, since $v_k = v_k(S_0, .., S_k)$ and $\delta_k = \delta_k(S_0, .., S_k)$ do not depend on S_{k+1}

$$\begin{aligned}
\mathsf{E}_{\text{rn}}\big[\, v_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] &= \mathsf{E}_{\text{rn}}\big[\, v_k + \delta_k(s_{k+1} - s_k) \,|\, \{S_j\}_{j=0}^{k} \,\big] \\
&= v_k + \delta_k \times \mathsf{E}_{\text{rn}}\big[\, s_{k+1} - s_k \,|\, \{S_j\}_{j=0}^{k} \,\big] \\
&= v_k + \delta_k \times \Big(\mathsf{E}_{\text{rn}}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] \;-\; s_k \Big) \\
&= v_k
\end{aligned}$$

where we used again the martingale property $\mathsf{E}_{\text{rn}}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] = s_k$ in the last line. ■

Remark: Equation (13) is actually equivalent to the recursion relation for the v_k's of Theorem 2.1, since the risk neutral probability is exactly given by the weight w_{up} of Theorem 2.1.