Week2b: Kapitel 2: Das Binomialmodell, Teil1

Let S be some tradable asset with prices

$$S_k \;=\; S(t_k), \quad k = 0, 1, 2, ... \tag{1}$$

and let

$$H \;=\; H(S_0, S_1, ..., S_{N-1}, S_N) \tag{2}$$

be some option payoff with start date t_0 and end date or maturity t_N. We want to replicate the option payoff (2) with a suitable trading strategy in the underlying S. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding δ_k assets at the end of day t_k generates the amount

$$V_N \;=\; V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{3}$$

Each δ_k will be determined on the end of trading day t_k. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, δ_k can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k \;=\; \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \tag{4}$$

Definition 2.1: We say that an option payoff (2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of δ_k of the form (4) and some initial amount V_0 such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_0 \;+\; \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{5}$$

The **initial amount $\mathbf{V_0}$** which is needed to set up the replicating strategy is called the theoretical fair value of H or **the price of the option H.** The process of replicating an option payoff H through formula (5), that is, through a trading strategy which holds δ_j pieces of the underlying S at the end of day t_j, is called **hedging**.

Now let us consider the question to what extent replication of option payoffs is possible. Equation (5) can be rewritten as

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$H(S_0,S_1,...,S_{N-1},S_N)-\delta_{N-1}\,S_N \;=\; V_{N-1}-\delta_{N-1}\,S_{N-1}$$
$$= \text{some function of } \; S_0,S_1,...,S_{N-1} \tag{6}$$

That is, the right hand side of (6) is independent of S_N. Let us introduce the return of the asset S from t_{k-1} to t_k,

$$\text{ret}_k \;:=\; \frac{S_k-S_{k-1}}{S_{k-1}} \;=\; \frac{S_k}{S_{k-1}} \;-\; 1 \tag{7}$$

such that

$$S_k \;=\; S_{k-1}(1+\text{ret}_k) \tag{8}$$

Then equation (6) can be rewritten as

$$H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\text{ret}_N)\big)-\delta_{N-1}\,S_{N-1}(1+\text{ret}_N) \;=\; \text{const} \tag{9}$$

where the const in the equation above means that the left hand side of (9) has to be the same for all possible choices of ret_N. Since there is only 1 free parameter in (9), namely δ_{N-1}, we can only allow for 2 possible choices for ret_N, say,

$$\text{ret}_N \;\in \{\text{ret}_{\text{up}},\text{ret}_{\text{down}}\} \tag{10}$$

and in that case we have to have

$$H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\text{ret}_{\text{up}})\big)-\delta_{N-1}\,S_{N-1}(1+\text{ret}_{\text{up}}) \;=$$
$$H\big(S_0,S_1,...,S_{N-1},S_{N-1}(1+\text{ret}_{\text{down}})\big)-\delta_{N-1}\,S_{N-1}(1+\text{ret}_{\text{down}})$$

which determines δ_{N-1} to

$$\delta_{N-1} \;=\; \frac{H\big(S_0,...,S_{N-1},S_{N-1}(1+\text{ret}_{\text{up}})\big)\;-\;H\big(S_0,...,S_{N-1},S_{N-1}(1+\text{ret}_{\text{down}})\big)}{S_{N-1}(1+\text{ret}_{\text{up}})\;-\;S_{N-1}(1+\text{ret}_{\text{down}})} \tag{11}$$

Thus, if we allow for 2 possible choices of returns only as in (10), replication of option payoffs seem to be possible. This leads to the following

Definition 2.2: If the price process $S_k=S(t_k)$ of some tradable asset S has the dynamics

$$S_k \;=\; S_{k-1}(\,1\,+\,\text{ret}_k\,) \qquad \text{with} \quad \text{ret}_k\in\{\text{ret}_{\text{up}},\text{ret}_{\text{down}}\} \tag{12}$$

for all k, then we say that S is given by the Binomial model.

Remark: Observe that in Definition 2.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability p_{up} such that an up-return ret_{up} will occur and a probability $p_{\text{down}}=1-p_{\text{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value V_0 are actually independent of such probabilities.

Now we are in a position to formulate the following important

Theorem 2.1: Let S be some tradable asset whose price process is given by the Binomial model (12). Let r denote some constant interest rate which is to be applied to cash amounts at each period from t_{k-1} to t_k. Then every option payoff

$$H \;=\; H(S_0, ..., S_N)$$

can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \;=\; \delta_k(S_0, \cdots, S_k) \;=\; \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{S_{k+1}^{\text{up}} - S_{k+1}^{\text{down}}} \tag{13}$$

with the abbreviations

$$\begin{aligned} S_{k+1}^{\text{up/down}} \;&:=\; S_k\,(\,1 + \text{ret}_{\text{up/down}}\,) \\ V_{k+1}^{\text{up/down}} \;&:=\; V_{k+1}\big(\,S_0\,,\,\cdots\,,\,S_k\,,\,S_{k+1}^{\text{up/down}}\,\big) \end{aligned}$$

and the portfolio values V_k, including the theoretical fair value, the option price V_0, can be inductively calculated through the following formulae: Let

$$R \;:=\; 1 + r$$

Then

$$V_k \;=\; R^k\, v_k$$

and the discounted portfolio values v_k can be calculated recursively through

$$v_k \;=\; w_{\text{up}}\, v_{k+1}^{\text{up}} \,+\, w_{\text{down}}\, v_{k+1}^{\text{down}} \tag{14}$$

and the recursion starts at $k = N$ with discounted portfolio values

$$v_N \;:=\; R^{-N}\, H(S_0, \cdots, S_N)$$

The weights w_{up} and w_{down} are given by

$$w_{\text{up}} \;=\; \frac{r - \text{ret}_{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \tag{15}$$

$$w_{\text{down}} \;=\; 1 - w_{\text{up}} \;=\; \frac{\text{ret}_{\text{up}} - r}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}\;. \tag{16}$$

Proof of Theorem 2.1: ..machen wir nächste Woche.

Wir machen noch ein erstes Beispiel an der Tafel wenn wir noch Zeit haben, wenn Sie das einmal gesehen haben, haben Sie ein ganz gutes Gefühl, wie das funktioniert. Wenn man nur so das Theorem sieht, wirkt das auf den ersten Blick vielleicht nicht zu erhellend.