Hochschule RheinMain
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Lösungen Übungsblatt 7
Finanzmathematik I

Aufgabe 1: Wir benutzen die Formel (2) von `Loesung6.pdf`, das war die folgende Formel:

$$\mathsf{E}\big[F(x_t)\big] \;=\; \int_{\mathbb{R}} F(\sqrt{t}\,v)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}}$$

Damit bekommen wir

$$\begin{aligned}
\mathsf{Prob}\big[\,|x_t| \,\geq\, \alpha\sqrt{t}\,\big] \;&=\; \int \chi\big(\,|x_t| \,\geq\, \alpha\sqrt{t}\,\big)\, dW(\{x_s\}_{0<s\leq T}) \\
&=\; \mathsf{E}\Big[\,\chi\big(\,|x_t| \,\geq\, \alpha\sqrt{t}\,\big)\Big] \\
&=\; \int_{\mathbb{R}} \chi\big(\,|\sqrt{t}\,v| \,\geq\, \alpha\sqrt{t}\,\big)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \\
&=\; \int_{\mathbb{R}} \chi\big(\,|v| \,\geq\, \alpha\,\big)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \\
&=\; \int_{-\infty}^{-\alpha} e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;+\; \int_{+\alpha}^{+\infty} e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \\
&=\; 2\int_{-\infty}^{-\alpha} e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \\
&=\; 2\,N(-\alpha)
\end{aligned}$$

Für $\alpha = 2$ liefert das

$$\mathsf{Prob}\big[\,|x_t| \,\geq\, 2\sqrt{t}\,\big] \;=\; 2\,N(-2) \;\approx\; 2\times 0.02275 \;=\; 4.55\%$$

Aufgabe 2: Wir benutzen das Theorem 4.1 für $m = 2$:

<u>Theorem 4.1</u>: *Let $F:\mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$. Then*

$$\int F(x_{t_1},...,x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) = \int_{\mathbb{R}^m} F(x_{t_1},...,x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \qquad (1)$$

where the p-functions on the right hand side of (1) are given by

$$p_\tau(x,y) \;:=\; \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}}\;.$$

Since all expectations and variances to be calculated depend on the Brownian motion observed at two times s and t with $s < t$, we have $m = 2$ for all integrals and

$$\mathsf{E}\big[F(x_s, x_t)\big] \;=\; \int_{\mathbb{R}^2} F(x_s, x_t)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \tag{2}$$

If the quantities to be calculated depend only on the difference $x_t - x_s$, that is, $F(x_s, x_t) = f(x_t - x_s)$ with some function f, we have furthermore

$$\begin{aligned}
\mathsf{E}\big[F(x_s, x_t)\big] &= \int_{\mathbb{R}^2} F(x_s, x_t)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(x_t - x_s)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(x_t - x_s)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(x_t - x_s)^2}{2(t-s)}}\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} f(y)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dx_s\, dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, dx_s \times \int_{\mathbb{R}} f(y)\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dy \\
&= 1 \times \int_{\mathbb{R}} f(\sqrt{t-s}\, v)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;=\; \int_{\mathbb{R}} f(\sqrt{t-s}\, v)\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}}
\end{aligned} \tag{3}$$

Thus, with the formulae from Übungsblatt5, we obtain:

2a)

$$\begin{aligned}
\mathsf{E}[\, x_t - x_s\,] &\overset{\text{Definiton of }\mathsf{E}[\,\cdot\,]}{=} \int (x_t - x_s)\, dW(\{x_t\}_{0<t\le T}) \\
&\overset{(3)}{=} \int_{\mathbb{R}} \sqrt{t-s}\, v\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;=\; 0
\end{aligned}$$

2b)

$$\begin{aligned}
\mathsf{E}[\,(x_t - x_s)^2\,] &= \int (x_t - x_s)^2\, dW(\{x_t\}_{0<t\le T}) \\
&\overset{(3)}{=} \int_{\mathbb{R}} \big(\sqrt{t-s}\, v\big)^2\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \\
&= (t-s) \int_{\mathbb{R}} v^2\, e^{-\frac{v^2}{2}}\, \tfrac{dv}{\sqrt{2\pi}} \;=\; t - s
\end{aligned}$$

which gives

$$\begin{aligned}
\mathsf{V}[\, x_t - x_s\,] &= \mathsf{E}[\,(x_t - x_s)^2\,] \;-\; \big(\mathsf{E}[\, x_t - x_s\,]\big)^2 \\
&= t - s
\end{aligned}$$

2c) Finally,

$$
\begin{aligned}
\mathsf{Cov}[\, x_s \,,\, x_t \,] &= \mathsf{E}\big[(x_s - \mathsf{E}[x_s])(x_t - \mathsf{E}[x_t])\big] \\
&= \mathsf{E}\big[(x_s - 0)(x_t - 0)\big] \\
&= \mathsf{E}\big[x_s(x_t - x_s + x_s)\big] \\
&= \mathsf{E}\big[x_s(x_t - x_s)\big] \;+\; \mathsf{E}\big[x_s^2\big] \\
&= \quad 0 \quad\quad + \quad s \quad = \quad s
\end{aligned}
$$

since

$$
\begin{aligned}
\mathsf{E}\big[x_s(x_t - x_s)\big] &\overset{(2)}{=} \int_{\mathbb{R}^2} x_s(x_t - x_s)\, p_t(0, x_s)\, p_t(x_s, x_t)\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} x_s(x_t - x_s)\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(x_t - x_s)^2}{2(t-s)}}\, dx_s\, dx_t \\
&= \int_{\mathbb{R}^2} x_s\, y\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dx_s\, dy \\
&= \int_{\mathbb{R}} x_s\, \frac{1}{\sqrt{2\pi s}}\, e^{-\frac{x_s^2}{2s}}\, dx_s \times \int_{\mathbb{R}} y\, \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{y^2}{2(t-s)}}\, dy \\
&= 0 \times 0 \;=\; 0
\end{aligned}
$$

and $\mathsf{E}\big[x_s^2\big] = s$ has been calculated in 2b.