Hochschule RheinMain WS 2021/22
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Lösungen Übungsblatt 5
Finanzmathematik I

Aufgabe 1: Gaussian Integrals:

a) $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \;=\; \sqrt{2\pi}$.

Proof a) With polar coordinates

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}$$

and

$$dx\,dy \;=\; r\,dr\,d\varphi$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx &= \left\{ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \ \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right\}^{1/2} \\ &= \left\{ \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dx\,dy \right\}^{1/2} \\ &= \left\{ \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr\,d\varphi \right\}^{1/2} \\ &= \left\{ 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r\,dr \right\}^{1/2} \\ &= \left\{ 2\pi \left(e^{-\frac{r^2}{2}}\right)\Big|_0^{\infty} \right\}^{1/2} \\ &= \sqrt{2\pi}\,. \end{aligned}$$

b) $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} \;=\; 1$.

Proof b) Follows directly from part (a) and the substitution of variables

$$y \;=\; \frac{x-\mu}{\sigma} \quad\Rightarrow\quad dy \;=\; \frac{dx}{\sigma}$$

since

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &\overset{(a)}{=} 1\,. \end{aligned}$$

c-f) Offensichtlich sind die Formeln (c),(d) und (e) Spezialfälle von (f),

$$\int_{-\infty}^{\infty} x^n\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \;=\; \begin{cases} (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

mit $(n-1)!! = (n-1)(n-3)(n-5)\cdots 3\cdot 1$.

Beweis f) With partial integration, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} x^n\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} &= \int_{-\infty}^{\infty} \underbrace{x^{n-1}}_{=f} \times \underbrace{x\, e^{-\frac{x^2}{2}}}_{=g'} \frac{dx}{\sqrt{2\pi}} \\ &= x^{n-1}(-e^{-\frac{x^2}{2}})\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (n-1)x^{n-2} \left(-e^{-\frac{x^2}{2}}\right) \frac{dx}{\sqrt{2\pi}} \\ &= 0 \,+\, (n-1) \int_{-\infty}^{\infty} x^{n-2}\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

Thus the exponent n has decreased by 2. Repeating this procedure, we get

$$\begin{aligned} \int_{-\infty}^{\infty} x^n\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} &= (n-1)(n-3)\cdots \begin{cases} 4\cdot 2 \times \int_{-\infty}^{\infty} x\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} & \text{if } n \text{ is odd} \\ 3\cdot 1 \times \int_{-\infty}^{\infty} 1\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} & \text{if } n \text{ is even} \end{cases} \\ &= (n-1)(n-3)\cdots \begin{cases} 4\cdot 2 \times 0 & \text{if } n \text{ is odd} \\ 3\cdot 1 \times 1 & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)!! & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

g) $\int_{-\infty}^{\infty} e^{\lambda x}\, e^{-\alpha\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \;=\; \frac{1}{\sqrt{\alpha}}\, e^{\frac{1}{\alpha}\frac{\lambda^2}{2}}\,, \qquad \lambda \in \mathbb{R},\ \alpha > 0\,.$

Proof g) By multiplying the above equation with $\sqrt{\alpha}\, e^{-\frac{1}{\alpha}\frac{\lambda^2}{2}}$ we have to show that

$$\sqrt{\alpha}\, e^{-\frac{1}{\alpha}\frac{\lambda^2}{2}} \int_{-\infty}^{\infty} e^{\lambda x}\, e^{-\alpha\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \;=\; 1\,.$$

This follows again from part (a), since the left hand side gives

$$\sqrt{\alpha} \int_{-\infty}^{\infty} e^{-\frac{1}{\alpha}\frac{\lambda^2}{2}} e^{\lambda x}\, e^{-\alpha\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \;=\; \sqrt{\alpha} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}(\lambda^2 - 2\alpha\lambda x + \alpha^2 x^2)} \frac{dx}{\sqrt{2\pi}}$$

$$
\begin{aligned}
&= \sqrt{\alpha}\int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}(\lambda-\alpha x)^2}\,\frac{dx}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{\alpha}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha}y^2}\,\frac{dy}{\sqrt{2\pi}} \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}\,\frac{dv}{\sqrt{2\pi}} \quad = \quad 1\,.
\end{aligned}
$$

Aufgabe 2: We have

$$
\begin{aligned}
p_s(x,y)p_t(y,z) &= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-(\frac{1}{2s}+\frac{1}{2t})y^2+(\frac{x}{s}+\frac{z}{t})y} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y^2-2\frac{xt+zs}{s+t}y)} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} e^{\frac{s+t}{2st}(\frac{xt+zs}{s+t})^2} \\
&= \frac{e^{-\frac{x^2}{2s}-\frac{z^2}{2t}}}{2\pi\sqrt{st}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} e^{\frac{(xt+zs)^2}{2st(s+t)}} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-x^2(\frac{1}{2s}-\frac{t}{2s(s+t)})-z^2(\frac{1}{2t}-\frac{s}{2t(s+t)})+\frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{x^2}{2(s+t)}-\frac{z^2}{2(s+t)}+\frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{\mathbb{R}} p_s(x,y)p_t(y,z)\,dy &= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \int_{\mathbb{R}} e^{-\frac{s+t}{2st}(y-\frac{xt+zs}{s+t})^2} dy \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \int_{\mathbb{R}} e^{-\frac{s+t}{2st}v^2} dv \\
&= \frac{1}{2\pi\sqrt{st}}\; e^{-\frac{(x-z)^2}{2(s+t)}}\; \sqrt{2\pi}\sqrt{\frac{st}{s+t}} \\
&= p_{s+t}(x,z)
\end{aligned}
$$

which proves the lemma.