Lösungen 4.Übungsblatt
Finanzmathematik I

Aufgabe 1: **a)** We have

$$S_N \;=\; S_0 \prod_{k=1}^{N} \bigl(1+\mathrm{ret}_k\bigr)$$

and, since all the returns are independent,

$$\begin{aligned}\mathsf{E}[\,S_N\,] \;&=\; \mathsf{E}\Bigl[S_0 \prod_{k=1}^{N} \bigl(1+\mathrm{ret}_k\bigr)\Bigr] \\ &=\; S_0 \prod_{k=1}^{N} \bigl(1+\mathsf{E}[\mathrm{ret}_k]\bigr) \;.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathsf{E}[\mathrm{ret}_k] \;&=\; +q\times p_{\mathrm{up}} \;+\; (-q)\times p_{\mathrm{down}} \\ &=\; q/2 - q/2 \;=\; 0\end{aligned}$$

Thus,

$$\mathsf{E}[\,S_N\,] \;=\; S_0 \prod_{k=1}^{N} \bigl(1+0\bigr) \;=\; S_0 \;.$$

b) The notation $\mathsf{E}\bigl[\,\cdots \bigm| \{S_j\}_{j=0}^{k}\,\bigr]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write

$$\begin{aligned}S_N \;&=\; S_0 \prod_{j=1}^{N} \bigl(1+\mathrm{ret}_j\bigr) \\ &=\; S_0 \prod_{j=1}^{k} \bigl(1+\mathrm{ret}_j\bigr) \prod_{j=k+1}^{N} \bigl(1+\mathrm{ret}_j\bigr) \\ &=\; S_k \prod_{j=k+1}^{N} \bigl(1+\mathrm{ret}_j\bigr)\end{aligned}$$

and obtain

$$\begin{aligned}\mathsf{E}\bigl[\,S_N \bigm| \{S_j\}_{j=0}^{k}\,\bigr] \;&=\; \mathsf{E}\Bigl[\,S_k \prod_{j=k+1}^{N} \bigl(1+\mathrm{ret}_j\bigr) \bigm| \{S_j\}_{j=0}^{k}\,\Bigr] \\ &=\; S_k \prod_{j=k+1}^{N} \bigl(1+\mathsf{E}[\mathrm{ret}_j]\bigr) \\ &=\; S_k \prod_{j=k+1}^{N} \bigl(1+0\bigr) \;=\; S_k \;.\end{aligned}$$

c) Because of part (a), we have

$$\begin{aligned} \mathsf{E}\Big[\frac{1}{N}\sum_{m=1}^{N} S_m\Big] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[S_m\big] \\ &= \frac{1}{N}\sum_{m=1}^{N} S_0 = S_0\ . \end{aligned}$$

d) Because of part (b), we obtain

$$\begin{aligned} \mathsf{E}\left[\frac{1}{N}\sum_{m=1}^{N} S_m \,\Big|\, \{S_j\}_{j=0}^{k}\right] &= \frac{1}{N}\sum_{m=1}^{N} \mathsf{E}\big[S_m \,\big|\, \{S_j\}_{j=0}^{k}\big] \\ &= \frac{1}{N}\left\{\sum_{m=1}^{k} \mathsf{E}\big[S_m \,\big|\, \{S_j\}_{j=0}^{k}\big] + \sum_{m=k+1}^{N} \mathsf{E}\big[S_m \,\big|\, \{S_j\}_{j=0}^{k}\big]\right\} \\ &= \frac{1}{N}\left\{\sum_{m=1}^{k} S_m + \sum_{m=k+1}^{N} S_k\right\} \\ &= \frac{k}{N} \times \frac{1}{k}\sum_{m=1}^{k} S_m + \frac{N-k}{N} \times S_k\ . \end{aligned}$$

e) This can be done in a similar way as part (a): Since

$$S_N = S_0 \prod_{k=1}^{N} \big(1 + \mathrm{ret}_k\big)$$

we have

$$\frac{S_0}{S_N} = \prod_{k=1}^{N} \frac{1}{1 + \mathrm{ret}_k}\ .$$

Since all the returns are independent,

$$\begin{aligned} \mathsf{E}\big[S_0/S_N\big] &= \mathsf{E}\left[\prod_{k=1}^{N} \frac{1}{1 + \mathrm{ret}_k}\right] \\ &= \prod_{k=1}^{N} \mathsf{E}\left[\frac{1}{1 + \mathrm{ret}_k}\right] \\ &= \prod_{k=1}^{N} \left\{\frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2}\right\} \\ &= \prod_{k=1}^{N} \left\{\frac{1}{1-q^2}\right\} \\ &= \frac{1}{(1-q^2)^N}\ . \end{aligned}$$

f) Again, the notation $\mathsf{E}\big[\cdots \,\big|\, \{S_j\}_{j=0}^{k}\big]$ means that the prices $S_1, S_2, ..., S_k$ are no longer stochastic, but they are given numbers, they have already realized. Thus, the expectation has to be taken only with respect to the returns $\mathrm{ret}_{k+1}, \mathrm{ret}_{k+2}, ..., \mathrm{ret}_N$. Therefore we write as in part (b)

$$S_0 \;/\; S_N \;=\; S_0 \;/\; \Big\{ S_k \prod_{j=k+1}^{N} \big(1+\mathrm{ret}_j\big) \Big\}$$

and obtain

$$\begin{aligned}
\mathsf{E}\big[\, S_0/S_N \,\big|\, \{S_j\}_{j=0}^k \big] &= S_0/S_k \;\mathsf{E}\Big[\prod_{m=k+1}^{N} \frac{1}{1+\mathrm{ret}_m} \,\Big|\, \{S_j\}_{j=0}^k \Big] \\
&= S_0/S_k \prod_{m=k+1}^{N} \mathsf{E}\Big[\frac{1}{1+\mathrm{ret}_m} \Big] \\
&= S_0/S_k \prod_{m=k+1}^{N} \Big\{ \frac{1}{1+q} \times \frac{1}{2} + \frac{1}{1-q} \times \frac{1}{2} \Big\} \\
&= S_0/S_k \prod_{m=k+1}^{N} \Big\{ \frac{1}{1-q^2} \Big\} \\
&= S_0/S_k \; \frac{1}{(1-q^2)^{N-k}} \;.
\end{aligned}$$

Aufgabe 2: If the price process $\{S_k\}_{k=0}^N$ should be a martingale, we must have

$$\mathsf{E}\big[\, S_k \,\big|\, \{S_j\}_{j=0}^{k-1} \big] \;=\; S_{k-1}$$

for all k. Now we have

$$\begin{aligned}
\mathsf{E}\big[\, S_k \,\big|\, \{S_j\}_{j=0}^{k-1} \big] &= \mathsf{E}\big[\, S_{k-1}(1+\mathrm{ret}_k) \,\big|\, \{S_j\}_{j=0}^{k-1} \big] \\
&= S_{k-1}\big(1+\mathsf{E}[\mathrm{ret}_k]\big) \\
&\overset{!}{=} S_{k-1}
\end{aligned}$$

which is the case if

$$\begin{aligned}
0 &\overset{!}{=} \mathsf{E}[\mathrm{ret}_k] \\
&= q \times p \;+\; \mathrm{ret}_{\mathrm{down}} \times (1-p)
\end{aligned}$$

which gives

$$\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{p}{1-p}\, q \;.$$

Thus, for $p = 1/4$ we get

$$\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{1/4}{3/4}\, q \;=\; -\frac{q}{3}$$

For $p = 1/2$, we get

$$\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{1/2}{1/2}\, q \;=\; -q$$

and for $p = 3/4$ we get

$$\mathrm{ret}_{\mathrm{down}} \;=\; -\frac{3/4}{1/4}\, q \;=\; -3q \;\;.$$