Lösungen Blatt 11
Finanzmathematik I

Aufgabe 1: **a)** We have

$$\begin{aligned} H_{\text{call,perf}}(S_T) &= \max\{\, S_T/S_0 - 1\,,\, 0\,\} \\ &= \tfrac{1}{S_0}\,\max\{\, S_T - S_0\,,\, 0\,\} \\ &=: \tfrac{1}{S_0}\, H_{\text{call,abs}}(S_T) \end{aligned}$$

where $H_{\text{call,abs}}(S_T) = H_{\text{call}}(S_T)$ is just a standard call in absolute amount to which the Black-Scholes formula can be applied. Thus, since $K = S_0$ and $r = 0$,

$$\begin{aligned} V_0 &= \text{price}\big(H_{\text{call,perf}}\big) \\ &= \tfrac{1}{S_0}\,\text{price}\big(H_{\text{call,abs}}\big) \\ &= \tfrac{1}{S_0}\,\big\{S_0\,N(d_+) \;-\; S_0\,N(d_-)\big\} \\ &= N(d_+) \;-\; N(d_-) \end{aligned}$$

with

$$\begin{aligned} d_\pm &= \frac{\log\big[\frac{S_0}{S_0}\big] + (0 \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \pm\,\frac{\sigma\sqrt{T}}{2}\;. \end{aligned}$$

This proves part (a).

b) Using (a), we can write

$$\begin{aligned} V_0 &= N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2) \\ &= \frac{N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2)}{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)} \times \big\{\sigma\sqrt{T}/2 - (-\sigma\sqrt{T}/2)\big\} \\ &\approx N'(0) \times \sigma\sqrt{T} \end{aligned}$$

Since

$$N(x) \;=\; \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$$

we have

$$N'(x) \;=\; e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}}$$

and we obtain

$$\begin{aligned} V_0 &\approx \frac{1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} \times \sigma\sqrt{T} \\ &= \frac{1}{\sqrt{2\pi}} \times \sigma\sqrt{T} \\ &= 0.398942.. \times \sigma\sqrt{T} \\ &\approx 0.4 \times \sigma\sqrt{T}\,. \end{aligned}$$

Aufgabe 2: a) Wir haben

$$\begin{aligned} V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H\Big(S_t\, e^{\sigma\sqrt{T-t}\,x \,+\, (r-\frac{\sigma^2}{2})(T-t)} \Big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} \left(\frac{S_t\, e^{\sigma\sqrt{T-t}\,x \,+\, (r-\frac{\sigma^2}{2})(T-t)}}{S_0} \right)^{\alpha} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{-r(T-t)} \left(\frac{S_t\, e^{\,(r-\frac{\sigma^2}{2})(T-t)}}{S_0} \right)^{\alpha} \int_{\mathbb{R}} e^{\alpha\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{(\alpha-1)r(T-t)}\, e^{\,-\alpha\frac{\sigma^2}{2}(T-t)} \left(\frac{S_t}{S_0} \right)^{\alpha} \int_{\mathbb{R}} e^{\alpha\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}}\, e^{-\frac{\alpha^2\sigma^2(T-t)}{2}} \frac{dx}{\sqrt{2\pi}}\, e^{+\frac{\alpha^2\sigma^2(T-t)}{2}} \\ &= e^{(\alpha-1)r(T-t)}\, e^{\,-\alpha\frac{\sigma^2}{2}(T-t)}\, e^{+\frac{\alpha^2\sigma^2(T-t)}{2}} \left(\frac{S_t}{S_0} \right)^{\alpha} \int_{\mathbb{R}} e^{-\frac{(x\,-\,\alpha\sigma\sqrt{T-t})^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= e^{(\alpha-1)r(T-t)}\, e^{\frac{(\alpha^2-\alpha)\sigma^2(T-t)}{2}} \left(\frac{S_t}{S_0} \right)^{\alpha} \end{aligned}$$

b) Mit dem Resultat aus Teil (a) ergibt sich mit $V_t = V(S_t, t)$

$$\frac{\partial V}{\partial S_t} \;=\; \frac{\alpha}{S_t}\, V$$

und

$$\frac{\partial^2 V}{\partial S_t^2} \;=\; \frac{\partial}{\partial S_t} \left(\frac{\alpha}{S_t}\, V \right) \;=\; -\,\frac{\alpha}{S_t^2}\, V \;+\; \frac{\alpha}{S_t}\, \frac{\alpha}{S_t}\, V$$

Also,

$$\begin{aligned} S_t\, \frac{\partial V}{\partial S_t} &= \alpha\, V \\ S_t^2\, \frac{\partial^2 V}{\partial S_t^2} &= -\,\alpha\, V \;+\; \alpha^2\, V \end{aligned}$$

Weiterhin,

$$\frac{\partial V}{\partial t} \;=\; \Big\{ -(\alpha-1)r \;-\; (\alpha^2-\alpha)\frac{\sigma^2}{2} \Big\}\, V$$

Also insgesamt,

$$\begin{aligned}
&\frac{\partial V}{\partial t} \;+\; \frac{\sigma^2}{2}S^2\,\frac{\partial^2 V}{\partial S^2} \;+\; rS\,\frac{\partial V}{\partial S} \;-\; rV \\
&\quad=\; \Big\{ -(\alpha-1)r \;-\; (\alpha^2-\alpha)\frac{\sigma^2}{2} \Big\}\, V \;+\; \frac{\sigma^2}{2}(\alpha^2-\alpha)V \;+\; r\alpha V \;-\; rV \\
&\quad=\; 0\ .
\end{aligned}$$