Lösungen 10. Übungsblatt
Finanzmathematik I

Aufgabe 1: Die allgemeine Pricing-Formel lautet

$$
\begin{aligned}
V_0 &= e^{-rT} \int_{\mathbb{R}} H(S_T)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}
\end{aligned}
$$

Wir müssen schauen, über welche x man tatsächlich integrieren muss:

$$
\begin{aligned}
& S_T \geq K \\
\Leftrightarrow\quad & S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T} \geq K \\
\Leftrightarrow\quad & \sigma\sqrt{T}\,x + (r-\sigma^2/2)T \geq \log[K/S_0] \\
\Leftrightarrow\quad & \sigma\sqrt{T}\,x \geq \log[K/S_0] - (r-\sigma^2/2)T \\
\Leftrightarrow\quad & x \geq \frac{\log[K/S_0] - (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\
\Leftrightarrow\quad & x \geq -\frac{\log[S_0/K] + (r-\sigma^2/2)T}{\sigma\sqrt{T}} \\
\Leftrightarrow\quad & x \geq -d_-
\end{aligned}
$$

Also:

$$
\begin{aligned}
V_0 &= e^{-rT} \int_{\mathbb{R}} H\big(S_0\, e^{\sigma\sqrt{T}\,x + (r-\sigma^2/2)T}\big)\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{-d_-}^{\infty} 1\, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT} \int_{-\infty}^{d_-} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= e^{-rT}\, N(d_-)\,.
\end{aligned}
$$

Aufgabe 2: Let us calculate more generally the price at some time $t \in [0,T]$. Then we have to use the following formula:

$$\begin{aligned}
V_t &= e^{-r(T-t)} \int_{\mathbb{R}} H\big(S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\sigma^2/2)(T-t)}\big)\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \int_{\mathbb{R}} \frac{S_0}{S_t\, e^{\sigma\sqrt{T-t}\,x + (r-\sigma^2/2)(T-t)}}\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \frac{S_0}{S_t} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x - (r-\sigma^2/2)(T-t)}\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \\
&= e^{-r(T-t)} \frac{S_0}{S_t} e^{-(r-\sigma^2/2)(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \\
&= e^{-2r(T-t)} \frac{S_0}{S_t} e^{\sigma^2/2(T-t)} \int_{\mathbb{R}} e^{-\sigma\sqrt{T-t}\,x}\, e^{-\frac{x^2}{2}}\, e^{-\frac{\sigma^2(T-t)}{2}}\, \frac{dx}{\sqrt{2\pi}}\, e^{\frac{\sigma^2(T-t)}{2}} \\
&= e^{(-2r+\sigma^2)(T-t)} \frac{S_0}{S_t} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^2}\, \frac{dx}{\sqrt{2\pi}} \\
&= e^{(-2r+\sigma^2)(T-t)}\; \frac{S_0}{S_t}\; \times 1 \\
&= \frac{S_0}{S_t}\, e^{(\sigma^2-2r)(T-t)}\; .
\end{aligned}$$

Aufgabe 3: Mit den Black-Scholes Formeln aus der Vorlesung bekommen wir

$$\begin{aligned}
d_\pm &= \frac{\log[S_0/K] \,+\, (r \pm \sigma^2/2)T}{\sigma\sqrt{T}} \\
&= \frac{\log[100/120] \,+\, (0.03 \pm (0.25)^2/2)4}{0.25 \times \sqrt{4}}
\end{aligned}$$

und

$$\begin{aligned}
d_+ &= 0.125357 \\
d_- &= -0.37464
\end{aligned}$$

In der Klausur gibt es eine Tabelle mit den $N(x)$-Werten, die haben dann nur 4 Stellen Genauigkeit. Aus Excel bekommt man etwas genauere Werte,

$$\begin{aligned}
N(d_+) &= 0.549879 \\
N(d_-) &= 0.353963
\end{aligned}$$

und damit dann

$$\begin{aligned}
V_0 &= S_0 N(d_+) - K e^{-rT} N(d_-) \\
&= 100 \times 0.549879 - 120\, e^{-0.12} \times 0.353963 \;=\; 17.3155\; .
\end{aligned}$$