Hochschule RheinMain | Finanzmathematik I
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Kapitel 4: Brownsche Bewegung, Wiener-Maß und das Black-Scholes Modell, Teil2

In der letzten Veranstaltung hatten wir uns zunächst an die Datenanalyse für die Returns

$$\mathrm{ret}_{t_k} \;=\; \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}$$

erinnert, die wir in der Excel/VBA-Vorlesung auf dem 9. und dem 10. Übungsblatt gemacht hatten. Dort hatten wir, rein durch Betrachten der Daten, schon die Gleichung für das zeitdiskrete Black-Scholes Modell hergeleitet,

$$S_{t_k} \;=\; S_{t_{k-1}}(1 + \mu\,\Delta t + \sigma\,\sqrt{\Delta t}\,\phi_k)$$

wobei die ϕ_k's standard-normalverteilte unabhängige Zufallszahlen sind. Dann hatten wir gesagt, dass wir Skalierungsverhalten mit Δt, also genauer, die Tatsache, dass man vor den Zufallszahlen ein $\sqrt{\Delta t}$ braucht und kein Δt, etwas genauer verstehen wollen, und hatten dazu den Ansatz

$$S_{t_k} \;=\; S_{t_{k-1}}(1 + \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\phi_k) \tag{1}$$

gemacht. Auf $\alpha = 1$ sind wir dann relativ schnell gekommen, und dann, um auf $\beta = 1/2$ zu kommen, hatten wir eine Rechnung angefangen und dazu erstmal das μ auf 0 gesetzt, $\mu = 0$. In dem Fall haben wir dann mit $N = N_t = t/\Delta t$ oder $t = t_N = N\Delta t$

$$S_t \;=\; S_0 \prod_{k=1}^{N_t} \bigl(1 + \sigma\,\Delta t^{\beta}\phi_k\bigr) \;=\; S_0\,\exp\Bigl\{\sum_{k=1}^{N_t} \log(1 + \sigma\,\Delta t^{\beta}\phi_k)\Bigr\}$$

Dann hatten wir die Taylor-Entwicklung

$$\log(1+x) \;=\; x \;-\; \frac{x^2}{2} \;+\; O(x^3)$$

benutzt, mit

$$x \;=\; \sigma\,\Delta t^{\beta}\phi_k$$

und waren auf

$$\begin{aligned} S_t \;&=\; S_0\,\exp\Bigl\{\sum_{k=1}^{N_t}\Bigl(\sigma\,\Delta t^{\beta}\phi_k \;-\; \frac{1}{2}\sigma^2\Delta t^{2\beta}\phi_k^2 \;+\; O(\Delta t^{3\beta})\Bigr)\Bigr\} \\ &=\; S_0\,\exp\Bigl\{\sigma\,\Delta t^{\beta}\sum_{k=1}^{N_t}\phi_k \;-\; \frac{\sigma^2}{2}\Delta t^{2\beta}\sum_{k=1}^{N_t}\phi_k^2 \;+\; O(\Delta t^{3\beta-1})\Bigr\} \end{aligned} \tag{2}$$

gekommen. Dieser Ausdruck, insbesondere die erste Summe im Exponenten,

$$\sigma\,\Delta t^{\beta} \sum_{k=1}^{N_t} \phi_k$$

legt es nahe, Erwartungswerte von der folgenden Form zu betrachten:

$$\mathsf{E}\Big[f\Big(\Delta t^{\beta} \sum_{k=1}^{N_t} \phi_k\Big)\Big] \;=\; \int_{\mathbb{R}^{N_t}} f\Big(\Delta t^{\beta} \sum_{k=1}^{N_t} \phi_k\Big) \prod_{k=1}^{N_t} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}}\, d\phi_k \tag{3}$$

Für festes $t = N\Delta t$, $N = N_t = t/\Delta t$, bekommt man da im Limes $\Delta t \to 0$ ein unendlich-dimensionales Integral, also es ist keineswegs klar, inwieweit man da einen sinnvollen Limes erhalten kann. Um das zu analysieren, hatten wir dann die folgende, sehr wichtige Variablensubstitution gemacht:

$$\begin{array}{rclcrcl} x_1 &=& \sqrt{\Delta t}\; \phi_1 & & \phi_1 &=& x_1/\sqrt{\Delta t} \\ x_2 &=& \sqrt{\Delta t}\; (\phi_1 + \phi_2) & & \phi_2 &=& (x_2 - x_1)/\sqrt{\Delta t} \\ x_3 &=& \sqrt{\Delta t}\; (\phi_1 + \phi_2 + \phi_3) & \Leftrightarrow & \phi_3 &=& (x_3 - x_2)/\sqrt{\Delta t} \\ &\vdots& & & &\vdots& \\ x_{N_t} &=& \sqrt{\Delta t}\; (\phi_1 + \phi_2 + ... + \phi_{N_t}) & & \phi_{N_t} &=& (x_{N_t} - x_{N_t-1})/\sqrt{\Delta t} \end{array} \tag{4}$$

Die Funktionaldeterminante, die man dann bei der Transformationsformel für N-dimensionale Integrale bekommt, war, mit $N = N_t$,

$$\det \frac{\partial \phi}{\partial x} \;=\; \frac{1}{\sqrt{\Delta t}^{\,N}}$$

Damit konnten wir dann die folgende Rechnung machen:

$$\begin{aligned} \mathsf{E}\Big[f\Big(\Delta t^{\beta} \sum_{k=1}^{N} \phi_k\Big)\Big] &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}} \sqrt{\Delta t} \sum_{k=1}^{N} \phi_k\Big) \prod_{k=1}^{N} \Big\{ \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{\phi_k^2}{2}} \Big\}\, d\phi_1 \cdots d\phi_N \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}}\, x_N\Big) \prod_{k=1}^{N} \Big\{ \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_k - x_{k-1})^2}{2\Delta t}} \Big\} \det\big[\tfrac{\partial \phi}{\partial x}\big]\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f\Big(\Delta t^{\beta-\frac{1}{2}}\, x_N\Big) \prod_{k=1}^{N} \Big\{ \tfrac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(x_{k-1} - x_k)^2}{2\Delta t}} \Big\}\, dx_1 \cdots dx_N \\ &= \int_{\mathbb{R}^N} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_N\big) \prod_{k=1}^{N} \Big\{ p_{\Delta t}(x_{k-1}, x_k)\, dx_k \Big\} \end{aligned} \tag{5}$$

where we introduced the kernel

$$p_\tau(x,y) := \tfrac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \tag{6}$$

and used the definition

$$x_0 := 0$$

The kernel (6) has the following basic property:

Lemma 4.1: *Let $p_t(x,y)$ be given by (6). Then*

$$\textstyle\int_{\mathbb{R}} p_s(x,y) p_t(y,z)\, dy \;=\; p_{s+t}(x,z) \tag{7}$$

Proof: Übungsblatt 5. ■

Using this lemma, we can actually perform the integrals over $x_1, x_2, \cdots, x_{N-1}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \cdots \int_{\mathbb{R}} dx_{N-1} \; \underbrace{p_{\Delta t}(x_0,x_1)\, p_{\Delta t}(x_1,x_2)}_{\int dx_1 \,\to\, p_{2\Delta t}(x_0,x_2)}\, p_{\Delta t}(x_2,x_3) \,\cdots\, p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_2 \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1} \; \underbrace{p_{2\Delta t}(x_0,x_2)\, p_{\Delta t}(x_2,x_3)}_{\int dx_2 \,\to\, p_{3\Delta t}(x_0,x_3)} \,\cdots\, p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_3 \cdots \int_{\mathbb{R}} dx_{N-1} \; p_{3\Delta t}(x_0,x_3) \,\cdots\, p_{\Delta t}(x_{N-1},x_N) \\
&= \int_{\mathbb{R}} dx_{N-1} \; p_{(N-1)\Delta t}(x_0,x_{N-1})\, p_{\Delta t}(x_{N-1},x_N) \\
&= p_{N\Delta t}(x_0,x_N)
\end{aligned}$$

Thus (5) simplifies to

$$\begin{aligned}
\mathsf{E}\Big[f\Big(\Delta t^{\beta} \textstyle\sum_{k=1}^{N_t} \phi_k\Big)\Big] &= \int_{\mathbb{R}^N} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_N\big) \prod_{k=1}^{N} \Big\{ p_{\Delta t}(x_{k-1},x_k)\, dx_k \Big\} && (8) \\
&= \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_N\big) \;\times\; p_{N\Delta t}(x_0,x_N)\, dx_N \\
&\overset{\substack{N\Delta t = t \\ x_0 = 0}}{=} \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\, x\big) \;\times\; \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\, dx && (9)
\end{aligned}$$

Hence, a nontrivial meaningful limit is obtained only for $\beta = \frac{1}{2}$.

Instead of labelling the x with $k \in \{1, 2, ..., N_t\}$, we label them with $t_k := k\Delta t$ which has the meaning of time. In particular, $t_N = N\Delta t = t$. So, we rename $x_k \to x_{k\Delta t} = x_{t_k}$. With that, we write down the following very important

Definition 4.1: *Let* $N_T = T/\Delta t$ *and* $p_t(x,y) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}}$. *Then the measure*

$$dW(\{x_t\}_{0<t\leq T}) \;:=\; \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{10}$$

is called the Wiener measure and the family of random variables or integration variables $\{x_t\}_{0<t\leq T}$ *is called a Brownian motion. In terms of i.i.d. random variables* $\phi_k \in \mathcal{N}(0,1)$,

$$x_t \;:=\; \lim_{\Delta t\to 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \phi_k \tag{11}$$

Remark: The time discretized version of the Wiener measure, this is what we actually will usually use, is simply given by a product of Gaussian normal distributions. With $N = N_T =$

$T/\Delta t$ and $t_k = k\Delta t$,

$$\begin{aligned} dW(\{x_{t_k}\}_{0<k\leq N}) &= \prod_{k=1}^{N} p_{\Delta t}(x_{t_{k-1}}, x_{t_k})\, dx_{t_k} \\ &= \prod_{k=1}^{N} \left\{ e^{-\frac{(x_{t_k}-x_{t_{k-1}})^2}{2\Delta t}} \frac{dx_{t_k}}{\sqrt{2\pi\Delta t}} \right\} \\ &= \prod_{k=1}^{N} \left\{ e^{-\frac{\phi_k^2}{2}} \frac{d\phi_k}{\sqrt{2\pi}} \right\} \end{aligned} \tag{12}$$

And the time discretized Brownian motion x_{t_k} is given by

$$x_{t_k} = \sqrt{\Delta t} \sum_{j=1}^{k} \phi_j \tag{13}$$

from which we get the recursion

$$x_{t_k} = x_{t_{k-1}} + \sqrt{\Delta t}\, \phi_k \tag{14}$$

The formulae (12,13,14) are very important and will be used over and over again. Integrals with respect to the Wiener measure are computed according to the following theorem, which also will be used over and over again.

Theorem 4.1: *Let $F : \mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \leq T$. Then*

$$\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) = \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \tag{15}$$

Proof: Because of (7) only the $x_{t_1}, ..., x_{t_m}$ integration variables survive:

$$\begin{aligned} &\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\leq T}) \qquad (16) \\ &= \lim_{\Delta t\to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \\ &= \lim_{\Delta t\to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_{t_1}} p_{\Delta t}(...)\, dx_{k\Delta t} \prod_{k=N_{t_1}+1}^{N_{t_2}} p_{\Delta t}(...)\, dx_{k\Delta t} \times \cdots \\ &\qquad\qquad \cdots \times \prod_{k=N_{t_{m-1}}+1}^{N_{t_m}} p_{\Delta t}(...)\, dx_{k\Delta t} \\ &\overset{(7)}{=} \lim_{\Delta t\to 0} \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m})\, p_{t_1}(x_0, x_{t_1})\, dx_{t_1}\; p_{t_2-t_1}(x_{t_1}, x_{t_2})\, dx_{t_2} \times \cdots \\ &\qquad\qquad \cdots \times\; p_{t_m - t_{m-1}}(x_{t_{m-1}}, x_{t_m})\, dx_{t_m} \\ &= \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \end{aligned}$$

which coincides with (15). ■

Now we return to (2) and put $\beta = \frac{1}{2}$. Then

$$S_t \;=\; S_0 \,\exp\left\{ \sigma \sqrt{\Delta t} \sum_{k=1}^{N_t} \phi_k \;-\; \frac{\sigma^2}{2} \Delta t \sum_{k=1}^{N_t} \phi_k^2 \;+\; O(\sqrt{\Delta t}) \right\} \tag{17}$$

The first term in the exponent converges to a Brownian motion $x_t = \lim_{\Delta t \to 0} \sqrt{\Delta t} \sum_{k=1}^{N_t} \phi_k$ and the last term vanishes, but what about the second term? There is the following

Theorem 4.2: Let

$$I_{\Delta t} \;:=\; \Delta t \sum_{k=1}^{N_t} \phi_k^2$$

with $N_t = t/\Delta t$ and $\phi_1, \phi_2, \cdots$ being independent Gaussian random numbers with mean 0 and standard deviation 1. Then the following statements hold:

a) $\mathsf{E}[I_{\Delta t}] \;=\; t$

b) $\mathsf{V}[I_{\Delta t}] \;=\; 2\,t\,\Delta t$

c) $\lim_{\Delta t \to 0} \mathsf{Prob}\Big[\, \big| I_{\Delta t} - t \big| \geq \varepsilon \,\Big] \;=\; 0 \qquad \forall \varepsilon > 0 \;.$

More intuitively, we may rewrite the statement of part (c) simply as

$$\lim_{\Delta t \to 0} I_{\Delta t} \;=\; \lim_{\Delta t \to 0} \Delta t \sum_{k=1}^{N_t} \phi_k^2 \;=\; t \;.$$

Proof: For standard normal distributed random numbers we have

$$\begin{aligned} \mathsf{E}[\phi^2] &= 1 \\ \mathsf{E}[\phi^4] &= 3 \\ \mathsf{V}[\phi^2] \;=\; \mathsf{E}[\phi^4] - \big(\mathsf{E}[\phi^2]\big)^2 &= 2 \end{aligned}$$

since more generally

$$\mathsf{E}[\phi^{2n}] \;=\; (2n-1)!!$$

Thus we get

$$\begin{aligned} \mathsf{E}[I_{\Delta t}] &= \Delta t \sum_{k=1}^{N_t} \mathsf{E}[\phi_k^2] \\ &= \Delta t \sum_{k=1}^{N_t} 1 \\ &= \Delta t \, N_t \;=\; t \;. \end{aligned}$$

To calculate the variance, we rewrite it as a covariance since the covariance is a bilinear quantity where we can bring sums from inside to the outside of the covariance as follows:

$$
\begin{aligned}
\mathsf{V}[I_{\Delta t}] &= \mathsf{Cov}[I_{\Delta t}, I_{\Delta t}] \\
&= \mathsf{Cov}\Big[\ \Delta t \sum_{k=1}^{N_t} \phi_k^2\ ,\ \Delta t \sum_{\ell=1}^{N_t} \phi_\ell^2\ \Big] \\
&= (\Delta t)^2 \sum_{k,\ell=1}^{N_t} \mathsf{Cov}[\ \phi_k^2\ ,\ \phi_\ell^2\ ] \\
&= (\Delta t)^2 \sum_{k,\ell=1}^{N_t} \delta_{k,\ell}\ \mathsf{Cov}[\ \phi_k^2\ ,\ \phi_k^2\ ] \\
&= (\Delta t)^2 \sum_{k=1}^{N_t}\ \mathsf{V}[\ \phi_k^2\ ] \\
&= (\Delta t)^2 \sum_{k=1}^{N_t}\ 2 \\
&= 2\,t\,\Delta t\ .
\end{aligned}
$$

Now we use Chebyshev's inequality. It states that for any random variable X we have

$$
\mathsf{Prob}\Big(\ \big|\,X\ -\ \mathsf{E}[X]\,\big|\ \geq\ \varepsilon\ \Big)\ \leq\ \frac{\mathsf{V}[X]}{\varepsilon^2}\ .
$$

Then we put $X = I_{\Delta t}$ such that with the results from part (a) and (b) we obtain

$$
\mathsf{Prob}\Big(\ \big|\,I_{\Delta t}\ -\ t\,\big|\ \geq\ \varepsilon\ \Big)\ \leq\ \frac{2t\Delta t}{\varepsilon^2}\ \stackrel{\Delta t\to 0}{\to}\ 0
$$

This proves the theorem. ■

We summarize our results: The statistics of financial data suggests, as a first approximation, the stochastic model (1). A meaningful continuous time model is only obtained if the exponents in (1) are choosen to be $\alpha = 1$ and $\beta = \frac{1}{2}$ which results in

$$
\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} = \frac{\Delta S_{t_k}}{S_{t_{k-1}}} = \mu\,\Delta t + \sigma\,\sqrt{\Delta t}\,\phi_k \tag{18}
$$

In view of (4), in particular, the right hand side thereof, and recalling the relabelling $x_k \to x_{k\Delta t}$, we may write this as

$$
\frac{\Delta S_{t_k}}{S_{t_{k-1}}} = \mu\,\Delta t + \sigma\,(x_{t_k} - x_{t_{k-1}}) = \mu\,\Delta t\ +\ \sigma\,\Delta x_{t_k} \tag{19}
$$

or, in the continuous time limit $\Delta t \to 0$,

$$
\frac{dS_t}{S_t} = \mu\,dt\ +\ \sigma\,dx_t \tag{20}
$$

where $\{x_t\}_{0<t\leq T}$ is a Brownian motion. And we saw that the discrete time solution (17) of (19) (for $\mu \neq 0$ there is an additional $\mu N_t \Delta t$ in the exponent, as we had in the previous lecture) converges to

$$S_t \;=\; S_0\, e^{\mu t + \sigma x_t - \frac{\sigma^2}{2} t} \tag{21}$$

The solution (21) is usually refered to as a 'geometric Brownian motion'. In a following section we will rederive (21) from (20) as an application of the Ito-Lemma. We have the following super-important

Definition 4.2: Let $\{x_t\}_{t\geq 0}$ be a Brownian motion. Then the stochastic process (21),

$$S_t \;=\; S_0\, e^{\mu t + \sigma x_t - \frac{\sigma^2}{2} t} \tag{22}$$

is called the **Black-Scholes model** for the asset price process $\{S_t\}_{t\geq 0}$. It is a solution of the stochastic differential equation (20),

$$\frac{dS_t}{S_t} \;=\; \mu\, dt \;+\; \sigma\, dx_t \tag{23}$$

Equation (23) is called the **Black-Scholes Stochastic Differential Equation** or Black-Scholes SDE (not to be confused with the Black-Scholes PDE, partial differential equation, which we discuss in Chapter 7).

In discrete time, Black-Scholes paths can be simulated through

$$S_{t_k} \;=\; S_{t_{k-1}} \left(1 \;+\; \mu\, \Delta t \;+\; \sigma \sqrt{\Delta t}\, \phi_k \right) \tag{24}$$

with the ϕ_k being standard normal distributed random numbers.

Excel/VBA-Demos:

- **a)** Show through simulation that for small Δt the S_{t_k}'s calculated iteratively through (24) and directly through (22) are approximately equal.
- **b)** Confirm part (c) of Theorem 4.2. That is, show through simulation that in discrete time for small Δt

$$I_{\Delta t} \;:=\; \Delta t \sum_{k=1}^{N_t} \phi_k^2 \;\approx\; t \;.$$